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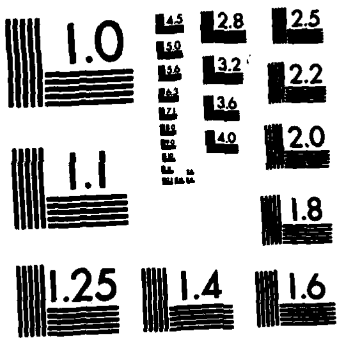
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A TRIDENT SCHOLAR PROJECT REPORT

NO. 121

AN ALGEBRAIC APPROACH TO EMPIRICAL SCIENCE AND QUANTUM LOGIC

UNITED STATES NAVAL ACADEMY
ANNAPOLIS, MARYLAND
1982

U.S.N.A. - Trident Scholar project report; no. 121 (1982)

AN ALGEBRAIC APPROACH TO EMPIRICAL SCIENCE AND QUANTUM LOGIC

A Trident Scholar Project Report

by

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[signature]

Chairman

29 July 82

Date

ABSTRACT

This paper develops some of the work of Foulis, Randall, Aerts, and Piron in the fields of empirical science and quantum logic from an algebraic point of view. More specifically, it begins with three axioms of what is called a "subtraction algebra," and generates various theorems associated with properties which are useful in empirical science.

After a foundation is established, it moves on to define the term "manual," a tool devised by Foulis and Randall in their study. ~~We define it~~ as a "dominated, atomic, semi-Boolean algebra" which satisfies an additional condition called "condition M." Several properties of the manual are discussed, and different types of manuals are given: classical semi-classical and non-classical.

~~We define~~ operational complements, operational perspectivity, atoms, events, and tests, before moving on to define a logic, and how it is derived from a manual. Properties of the logic are discussed, including a subtraction operation, a partial order, and an ortho complement.

Next, a computer program is presented. Its purpose is to take a finite semi-Boolean algebra and decide if the algebra is a manual. This is followed by a brief non-classical probabilistic discussion, which includes topices such as weights, pure states, and dispersion-free states.

Aerts' and Piron's work with properties, states, and questions is briefly discussed before moving on to several examples, some of them arising from navigation problems. The examples include the "hook," the "square," the "Wright Triangle," and the "free algebra." Empirical techniques are demonstrated on these examples. The examples comprise the bulk of this paper.

Preface

In regard to this paper, I owe thanks to many different people. Initially, I owe my thanks to the U. S. Naval Academy and the Trident Scholar Committee for giving me the opportunity to spend a year in undergraduate research. Very few individuals are afforded the privilege of knowing the rewards and frustations of such an endeavor.

In addition, I am deeply indebted to my advisor, Professor James C. Abbott, for having a vision for the project; and for his dedication, skill, guidance, and encouragement. Without him, the project would have lacked direction, and would not have been as challenging. Thanks are also due to Assistant Professor Steven Butcher and Assistant Professor Karen Zak, who attended the seminars weekly, offered a variety of valuable ideas, and always gave their encouragement and support.

Thanks go to Peter Haglich, who, as my fellow Trident Scholar, not only proved his mathematical brilliance, but his dedication, hard work, and friendship. His talents should carry him far. Thanks to Mrs. Ann Hardy, who took my chicken scratchings, and turned them into a clean, professional-looking report.

I also owe thanks, more than can be expressed, to my family and friends who encouraged me in my work, especially when the work seemed most frustrating. And finally, I owe all praise and glory to God who heard and answered my prayers while I was working on this paper.

Tim Thomas
Annapolis, Maryland

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INTRODUCTION

What began as a project designed to examine the present formal mathematical structure for quantum mechanics has turned out to be more of an examination of empirical logic, the science of interpreting outcomes of physical experiments. This is certainly a natural metamorphosis, since empirical logic enables one to interpret quantum mechanical results on a most fundamental level, allowing one to explore the interrelationship of outcomes of physical experiments without the constraints of some dominating mathematical structure.

What makes quantum mechanics an important area to study using empirical techniques is that quantum theory is so accurate -- it has been able to predict results of experiments to a high degree of accuracy. It is interesting to study quantum physics because the results are not the ones that an individual indoctrinated only in classical physics would expect to receive.

In classical physics, one deals in general with macroscopic bodies. These bodies exhibit properties which are natural to us, due to the fact that we observe classical systems in action every day. For example, we get a feel for the laws of conservation of momentum and conservation of energy every time we play a game of tennis. We can also identify things such as time, mass, position, speed, acceleration, and even energy every time we go for a ride in a car.

But when we discuss things in the realm of quantum physics, we are talking about what occurs on a microscopic level. We may deal with atoms, or even parts of atoms like electrons, protons, neutrons, or any one of the other particles discovered and added to the "particle zoo" collection. On this level, our conception of what is natural falls apart.

For example, referring back to the car example, while driving down the highway it would seem perfectly natural to identify both our position and momentum (in classical physics, the product of mass and velocity) simultaneously. All we have to do is note our position while glancing at the speedometer. However, in quantum physics, Heisenberg's Uncertainty Principle tells us that position and momentum cannot both be measured simultaneously with a high degree of accuracy. This inaccuracy is not due to the measuring techniques, but is due to the properties which are being measured.

We can measure either position or momentum, theoretically, to find the precise value for one or the other. But there is no universal test, no omnipotent test capable of reading both values precisely. Many times in areas outside of quantum physics, one finds instances in which an omnipotent test is missing.

In the chapter on examples, for instance, the reader will find that often in navigation a single lookout will not be able to observe 360 degrees around the ship. Thus, the captain must rely on reports from several lookouts, each with a limited field of vision. In this paper, we present different techniques for dealing with cases where there is no universal test.

Another example of a case in quantum physics where one gets results different from what one would expect in a classical situation comes in a variation of the Stern-Gerlach experiment. In this experiment one passes a particle through a magnetic field oriented along a given axis. The particle will deflect a given amount either up or down, depending upon the spin. For simplicity's sake, let us restrict ourselves to spin state +1/2 and -1/2. We will call +1/2 "spin up", and -1/2 "spin down."

Let us first pass the particle through an apparatus oriented in a given axis. We will call it the x-axis. Let us assume the particle is spin up. Let us now take the particle and pass it through another apparatus oriented in a direction rotated 90 degrees from the x-axis. We will call this the y-axis. Since the orientation is different, we could get either spin up or spin down. Again, let us assume that it is spin up. If we pass it through another apparatus oriented in the x-direction, we would expect that it would be spin up, since our apparati are set up simply as measuring devices. However, as it turns out, the particle could either be spin up or spin down, a totally non-classical result!

One may ask, "Why do we want to change or challenge a theory which would predict that unexpected outcome to the variation of the Stern-Gerlach experiment?" The answer is simply that we do not necesssarily want to change the theory, but be able to explain it from a rigorous mathematical approach. For throughout the evolution of quantum physics, challenges have been made to the propriety of many assumptions.

In the 1930's, Birkhoff and Von Neumann proposed that quantum theory should use orthomoslar lattice theory along with Hilbert spaces, which are essentially infinite dimensional inner product spaces. Later, others suggested that quantum theory need not be based on Hilbert spaces at all. They argue that Hilbert spaces were used in the beginning since they were the only infinite dimensional structures developed at the time. In 1956, Mackey attempted to set down the axioms of quantum logics, but as of today, the axioms are still incomplete.

The reader can see, then, that there are many unanswered questions in terms of a rigorous mathematical development of quantum physics. Quantum logicains, empirical scientists concentrating in quantum physics, attempt to reexamine a field which was first developed from a pragmatic viewpoint, and develop it from a logical viewpoint.

That is why we turn to examine logic. The first formalization of logic took place when George Boole formalized symbolic logic, in which he established an algebra of logic. Many are familiar with results of Boole's work, for all of these results contain a distributive property, a feature which again is natural to us, for we can observe it in the real world.

Huntington expanded on Boole's work,and as a result, postulated axioms for Boolean algebras. In 1936, Stone went one step further to demonstrate the relationship between Boolean algebras and algebras of sets.

In 1960, Kleindorfer and Abbott, working at the Naval Academy, investigated the implication connective in a formal logic. Since then Trident scholars and Trident-like scholars have built on their work and developed implication algebras. These include Academy graduates such as Pilling, Kelsy, Heard, and Kimble. Implication algebras are essentially subtraction algebras, except the symbology and order of writing the variables are different. The reader will find in Chapters 1 and 2 that subtraction algebras are important in our development of empirical science. This importance is due to their relationship to semi-Boolean algebras, which form the main structure in the development of empirical science by the Foulis-Randall school.

Foulis and Randall have asserted that empirical science must arise from physical observables -- results of tests. They also believe that empirical science should be free of dependence on explanations of physical events, and that each test should admit "the Boolean notions of conjunction, disjunction, negation, and so forth."[1] This axiom, then, explains why semi-Boolean algebras are so important.

A large portion of this paper follows the work of Foulis and Randall, and certainly none of the work violates the axioms just stated. This paper is set up in such a way that the theory and applications are separated: the first six chapters are theoretical, and Chapter 7 is one of examples and explanations.

Chapter 1 is a chapter which establishes various properties of subtraction algebras. Some of this work is similar to work previously done by Naval Academy midshipmen.

Chapter 2 is one which establishes the equivalence of subtraction algebras and semi-Boolean algebras. It goes on to discuss a special type of semi-Boolean algebra called a manual, which Foulis and Randall designed as a tool in empirical analysis. The term "manual" comes from describing a manual of instructions which tells what tests may be performed, and what outcomes to these tests will be allowed.

Many other terms in this chapter, although rigorously defined mathematically, may lack motivation. An "atom" or "outcome" is simply an outcome to a test. Make a test, get an outcome. The idea is that an atom is an event which cannot be further decomposed into simpler sub-events. A "test" or "operation" is simply what it says. And in practice, one writes a test as the set of outcomes allowed as a result of performing that test. An "event" is something that happened as a result of a test, and is made up of legal outcomes to that test. That is, an event is any set of outcomes resulting from a single test. Hence, each outcome is itself an event. At the other extreme, a test is itself a maximal event, the set of all possible outcomes of some fixed test.

An "operational complement" is simply a complement to an event with respect

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[1] Foulis and Randall, "What are Quantum Logics, and what ought they to be?", from the Proceedings of the Workshop on Quantum Logic, Ettore Majorana Centre for Scientific Culture, Ettore, Sicily, Dec. 2-9, 1979, pp. 9-10.

to the operation in which it is contained. Since a single event may occur as a result of distinct tests, it is possible for distinct events to to be operational complements of a single event, taken with respect to differnt tests. Hence, it is natural to call two events "operationally perspective" if they share a common operational complement. Thus, if one event occurs, its operational complement will non-occur, so that in turn an operationally perspective event will occur. In this sense, operationally perspective means that one event is essentially identical in the physical world to another event, regardless of the operations in which either is contained.

An event is said to "occur" if when a test is perfomed, the event results with a probability of 1. On the other hand, an event is said to "non-occur" if when a test is performed, the probability of the event occurring is 0. So, the reader may see now that these terms are motivated by the physical world, and yet will note in this chapter that they are only defined from a mathematician's perspective.

One further note from this chapter is that "DASBAM" is simply shorthand notation for dominated, atomic, semi-Boolean algebra satisfying condition M. Dominated, in common terms, means that there is a set of maximal elements which are exactly the tests. Atomic means that there is a set of atoms (outcomes). Sometimes we talk about a "DASBA" instead of a "DASBAM." The reader can probably surmise its meaning.

In Chapter 3 we discuss logics. Logics, very simply, are manuals with op pairs fused together -- considered as a single result. They are called logics since we can show that in a manual op is an equivalence relation, and as a result, the logic is comprised of equivalence classes.

In Chapter 4, we look at a particular case of Stone's Representation Theorem, which allows us to view the manuals as either Boolean algebras within each test, or as algebras of sets. This also allows us the freedom of notation necessary in constructing a computer program which will check condition M. This program is useful because it saves much tedium in verifying that some large semi-Boolean algebras are manuals.

Chapter 5 suggests another use of the computer in empirical science. After defining "weights" or probabilities on a manual, we seek to find all allowable states. Through a linear programming technique, we can find the extreme points of the set of allowable states. This technique is easily adaptable to the computer.

Finally, in the last chapter of theory, Chapter 6, we switch momentarily from the Foulis-Randall school to examine some of the work of Aerts and Piron of the Institute of Theoretical Physics in Geneva, Switzerland. Though initially their work will seem very different, we will see in the examples that their work is actually very closely related to the work of Foulis and Randall.

In the last chapter, we put together several examples to illustrate the concepts discussed in the first six chapters. Some of these examples deal with navigation problems, in which several lookouts are stationed in such a way that not one can see in an entire 360 degree field of view.

This is just one possible application of empirical science in a traditionally classical problem.

Many other areas appear to be fertile for application of empirical science. Due to the time constraints, though, we have limited the development to navigation problems. But one point that needs to be made is that though empirical science has been developed primarily for interpreting quantum physics, in this paper we apply it to problems encountered outside this realm.

The reader is challenged while reading this paper to consider possible applications to a field of study of interest to himself.

CHAPTER 1: Subtraction Algebras

Perhaps the best way to begin this discussion of empirical science is to start with the "machinery" - the foundations which allow us to be certain of the mathematical correctness of our methods. Most fundamental of the machinery is a background in subtraction algebra.[1,2] So, this is where the discussion begins.

Defn. 1.1 A subtraction algebra is a set S with an operation "$\setminus$" $\ni \forall x,y,z \in S$, the following three axioms hold:

S1) $x\setminus(y\setminus x) = x$

S2) $x\setminus(x\setminus y) = y\setminus(y\setminus x)$

S3) $(z\setminus y)\setminus x = (z\setminus x)\setminus y$

Thm. 1.1 $x, y \in S$, $(x\setminus y)\setminus y = x\setminus y$

Pf. $(x\setminus y)\setminus y = (x\setminus y)\setminus(y\setminus(x\setminus y))$ by S1
$= x\setminus y$ by S1

Thm. 1.2 $x, y \in S$, $x\setminus x = (y\setminus x)\setminus(y\setminus x)$

Pf. $x\setminus x = (x\setminus(y\setminus x))\setminus(x\setminus(y\setminus x))$ by S1
$= (x\setminus(x\setminus(y\setminus x)))\setminus(y\setminus x)$ by S3
$= ((y\setminus x)\setminus((y\setminus x)\setminus x))\setminus(y\setminus x)$ by S2
$= ((y\setminus x)\setminus(y\setminus x))\setminus(y\setminus x)$ by Thm. 1.1
$= (y\setminus x)\setminus(y\setminus x)$ by Thm. 1.1

Thm. 1.3 $x, y \in S$, $x\setminus x = y\setminus y$

Pf. $x\setminus x = (y\setminus x)\setminus(y\setminus x)$ by Thm. 1.2
$= (y\setminus(y\setminus x))\setminus(y\setminus(y\setminus x))$ by Thm. 1.2
$= (x\setminus(x\setminus y))\setminus(x\setminus(x\setminus y))$ by S2
$= (x\setminus y)\setminus(x\setminus y)$ by Thm. 1.2
$= y\setminus y$ by Thm. 1.2

Defn. 1.2 $\exists\, o \in S \ni \forall x \in S$, $x\setminus x = o$

Thm. 1.4 $\forall x \in S$, $x\setminus o = x$ and $o\setminus x = o$

Pf. $x\setminus o = x\setminus(x\setminus x)$ by Defn. 1.2
$= x$ by S1
$o\setminus x = o\setminus(x\setminus o)$ by first part of Thm. 1.4
$= o$ by S1

Thm. 1.5 $\forall x, y \in S$, $(x\setminus y)\setminus(y\setminus x) = x\setminus y$

Pf. $(x\setminus y)\setminus(y\setminus x) = (x\setminus(y\setminus x))\setminus y$ by S3
$= x\setminus y$ by S1

Defn. 1.3 $\forall x, y \in S$, $x \leq y$ iff $x\setminus y = o$

Thm. 1.6 If $a, x \in S$, $a \setminus x = o$, $x \setminus a = x$, then $a = o$

Pf. $a = a \setminus o$ by Thm. 1.4
$= a \setminus (a \setminus x) = x \setminus (x \setminus a)$ by S2
$= x \setminus x$
$= o$ by Defn. 1.2

Thm. 1.7 $\forall\, x, y \in S$ then $(x \setminus y) \setminus x = o$

Pf. $(x \setminus y) \setminus x = (x \setminus x) \setminus y$ by S3
$= o \setminus y$ by Defn. 1.2
$= o$ by Thm. 1.4

Defn. 1.4 $\forall\, x, y \in S$, $x \wedge y = x \setminus (x \setminus y)$. This is pronounced "x meet y."

Thm. 1.8 $\forall\, x, y \in S$, $x \leq y$ iff $x \wedge y = x$

Pf. Let $x \leq y$. Then $x \wedge y = x \setminus (x \setminus y)$ by Defn. 1.4. $x \setminus y = o$ by Defn. 1.3. Thus, $x \wedge y = x \setminus (x \setminus y) = x \setminus o = x$
Let $x \wedge y = x$. Then $x \setminus (x \setminus y) = x$ by Defn. 1.4. By Theorems 1.6 and 1.7, $x \setminus y = o$. By Defn. 1.3, $x \leq y$. Therefore, $x \leq y$ iff $x \wedge y = x$ iff $x \setminus y = o$.

Thm. 1.9 $y \setminus x = o$ implies that $\forall\, z \in S$, $(z \setminus x) \setminus (z \setminus y) = o$

Pf. Let $y \setminus x = o$, $z \in S$. $(z \setminus x) \setminus (z \setminus y) = (z \setminus (z \setminus y)) \setminus x$ by S3
$= (y \setminus (y \setminus z)) \setminus x$ by S2
$= (y \setminus x) \setminus (y \setminus z)$ by S3
$= o \setminus (y \setminus z) = o$ by Thm. 1.4

Thm. 1.10 $\forall\, x, y \in S$, then $x \setminus y = y \setminus x$ implies that $x = y$.

Pf. $x = x \setminus (y \setminus x)$ by S1
$= x \setminus (x \setminus y) = y \setminus (y \setminus x)$ by S2
$= y \setminus (x \setminus y) = y$ by S1

Thm. 1.11 "$\leq$" is a partial order.

Pf. (i) We will show that $x \leq x$. $x \leq x$ iff $x \setminus x = o$, which is true by Defn. 1.2.

(ii) Next we will show that $x \leq y$, $y \leq x$ implies that $x = y$. Recall that $x \leq y$, $y \leq x$ means that $x \setminus y = o = y \setminus x$ by Defn. 1.3. Thm. 1.10 proves that $x = y$.

(iii) Finally, we need to show that $x \leq y$ and $y \leq z$ implies that $x \leq z$. $y \leq z$ implies that $o = x \setminus y \geq x \setminus z$ by Thm. 1.8. $x \setminus z \geq o$ by Thm. 1.4, which actually says that $\forall\, x \in S$, $x \geq o$. Therefore, $o \leq x \setminus z \leq o$, which by part (ii) above implies that $x \setminus z = o$. By Defn. 1.3, $x \leq z$.

Thm. 1.12 $z \leq x$ iff $\exists\, y \in S \ni z = x \setminus y$

Pf. Let $z \leq x$. $x \setminus (x \setminus z) = z \setminus (z \setminus x)$ by S2
$= z \setminus o$ by Defn. 1.3
$= z$ by Thm. 1.4

Let $z = x \setminus y$ for some $y \in S$. Then $z = x \setminus y \leq x$ by Thm. 1.7.

<u>Thm. 1.13</u> $x \setminus (x \setminus (x \setminus y)) = x \setminus y \quad x, y \in S$

<u>Pf.</u> $x \setminus (x \setminus (x \setminus y)) = (x \setminus y) \setminus ((x \setminus y) \setminus x)$ by S2
$= (x \setminus ((x \setminus y) \setminus x)) \setminus y$ by S3
$= x \setminus y$ by Thm. 1.7.

<u>Thm. 1.14</u> $x \wedge y$ is the greatest lower bound of x and y with respect to $\leq$.

<u>Pf.</u> Let z be a lower bound for x and y. Show $z \leq x \wedge y$. $z \leq x$ implies that $z \setminus x = 0$, and $x \setminus (x \setminus z) = z$. $z \leq y$ implies that $z \setminus y = o$, and $y \setminus (y \setminus z) = z$.
$z \leq x \wedge y$ iff $z \setminus (x \wedge y) = o$. $z \setminus (x \wedge y) = z \setminus (x \setminus (x \setminus y))$ by Defn. 1.4.
$= (x \setminus (x \setminus z)) \setminus (x \setminus (x \setminus y))$ by S1
$= (x \setminus (x \setminus (x \setminus y))) \setminus (x \setminus z)$ by S3
$= (x \setminus y) \setminus (x \setminus z)$ by Thm. 1.13
$= (x \setminus (x \setminus z)) \setminus y$ by S3
$= z \setminus y = o$

Therefore, $z \leq x \wedge y$, $\forall z \leq x, y$. $x \wedge y$ is the greatest lower bound for x and y.

<u>Thm. 1.15</u> $x \setminus y = x$ iff $x \wedge y = o$ iff $y \setminus x = y$

<u>Pf.</u> Let $x \setminus y = x$. Then $x \setminus (x \setminus y) = o = x \wedge y$ by Definitions 1.3 and 1.4. Let $x \wedge y = o$. Then $x \wedge y = o = y \setminus (y \setminus x)$, again by Definitions 1.3 and 1.4. Then since $y \geq y \setminus x \geq y$, $y = y \setminus x$. Let $y \setminus x = y$. Then $o = y \setminus (y \setminus x) = x \setminus (x \setminus y)$ by S2, which implies that $x \leq x \setminus y \leq x$. Therefore $x \setminus y = x$.

<u>Thm. 1.16</u> (OMS - orthomodular law for subtraction) $\forall$ x, y, a $\ni x \leq y \leq a$, then $x = y \setminus (a \setminus x)$

<u>Pf.</u> $y \setminus (a \setminus x) = (y \setminus (y \setminus a)) \setminus (a \setminus x)$ by S2
$= (a \setminus (a \setminus y)) \setminus (a \setminus x)$ by S2
$= (a \setminus (a \setminus x)) \setminus (a \setminus y)$ by S3
$= (x \setminus (x \setminus a)) \setminus (a \setminus y)$ by S2
$= x \setminus (a \setminus y)$

$x \leq y$ implies that $a \setminus x \geq a \setminus y$, by Thm. 1.9. Using the same theorem,
$x = x \setminus (a \setminus x) \leq x \setminus (a \setminus y)$
Since $x \leq x \setminus (a \setminus y) \leq x$, $x = x \setminus (a \setminus y) = y \setminus (a \setminus x)$

<u>Thm. 1.17</u> For $z \leq x$ and $z, x \in S$, then $z \setminus (y \setminus x) = z$.

<u>Pf.</u> $z \leq x$ implies that $y \setminus z \geq y \setminus x$, which in turn implies that
$z = z \setminus (y \setminus z) \leq z \setminus (y \setminus x) \leq z$.
Therefore, $z = z \setminus (y \setminus x)$

<u>Thm. 1.18</u> For $z \leq x$ and $z, x \in S$, then $(y \setminus z) \setminus x = y \setminus x$

<u>Pf.</u> $(y \setminus z) \setminus x = (y \setminus x) \setminus z$ by S3, which says that $(y \setminus z) \setminus x \leq y \setminus x$.
We need to show that $y \setminus x \leq (y \setminus z) \setminus x$. $(y \setminus x) \setminus ((y \setminus z) \setminus x) = (y \setminus x) \setminus ((y \setminus x) \setminus z)$
$= z \setminus (z \setminus (y \setminus x))$ by S3 and S2.
$= z \setminus z$ by Thm. 1.17
$= o$. Therefore, $(y \setminus z) \setminus x = y \setminus x$.

Thm. 1.19 Let $z \leq x \leq y$, $x, y, z \in S$. Then $(y \setminus z) \setminus (x \setminus z) = y \setminus x$.

Pf.
$$\begin{aligned}(y \setminus z) \setminus (x \setminus z) &= (y \setminus (x \setminus z)) \setminus z \text{ by S3} \\ &= (y \setminus (x \setminus (x \setminus (y \setminus z)))) \setminus z \text{ by Thm. 1.16} \\ &= (y \setminus ((y \setminus z) \setminus ((y \setminus z) \setminus x))) \setminus z \text{ by S2} \\ &= (y \setminus ((y \setminus z) \setminus (y \setminus x))) \setminus z \text{ by Thm. 1.18} \\ &= (y \setminus z) \setminus ((y \setminus z) \setminus (y \setminus x)) \text{ by S3} \\ &= (y \setminus z) \setminus (y \setminus x) \text{ by Defn. 1.4} \\ &= y \setminus x \text{ by Thm. 1.7.}\end{aligned}$$

Defn. 1.5 If $x \leq a$, we define the relative complement of x with respect to a, $x_a^{\perp} = a \setminus x$.

As a result of this definition, Thm. 1.13 can now be read
$x_a^{\perp} = ((x_a^{\perp})_a^{\perp})_a^{\perp}$

Thm. 1.20 (compatibility) $x \leq y \leq a$ implies that $x_y^{\perp} = x_a^{\perp} \wedge y$.

Pf. $x_a^{\perp} \wedge y = (a \setminus x) \wedge y = y \setminus (y \setminus (a \setminus x)) = y \setminus x$ by Thm. 1.16.
$= x_y^{\perp}$.

Another property we would expect of the relative complement is that the meet of x and its relative complement would be o.

Thm. 1.21 $x \wedge x_a^{\perp} = o$

Pf. $x \wedge x_a^{\perp} = x \wedge (a \setminus x) = x \setminus (x \setminus (a \setminus x)) = x \setminus x = o$

Defn. 1.6 Let $x, y \leq e$, $\ni$ $x, y, e \in S$. We define $x \vee y = e \setminus ((e \setminus x) \setminus (y \setminus x))$.
$x \vee y$ is pronounced "x join y."

Note that both x and y must be less than a single element in the set S in order for the join to exist. In general, the join of two arbitrary elements does not exist.

Thm. 1.22 If $\exists e \in S$ $\ni$ $x, y \leq e$, then $x \vee y$ is an upper bound for x and y.

Pf. If $x \vee y$ is an upper bound for x and y, then $x, y \leq x \vee y$. By Theorem 1.8, $x \leq x \vee y$
iff $x \wedge (x \vee y) = x$.
$$\begin{aligned}x \wedge (x \vee y) &= x \wedge (e \setminus ((e \setminus x) \setminus (y \setminus x))) \\ &= (e \setminus ((e \setminus x) \setminus (y \setminus x))) \setminus ((e \setminus ((e \setminus x) \setminus (y \setminus x))) \setminus x) \\ &\quad \text{by Defn. 1.4} \\ &= (e \setminus ((e \setminus x) \setminus (y \setminus x))) \setminus ((e \setminus x) \setminus ((e \setminus x) \setminus (y \setminus x))) \\ &\quad \text{by S3} \\ &= e \setminus (e \setminus x) \text{ by Thm. 1.19} \\ &= x \quad \text{by Defn. 1.4 and the assumption} \\ &\quad \text{that } x \leq e\end{aligned}$$

Thus, we have shown $x \leq x \vee y$. We must show $y \leq x \vee y$.

But note the following:

$$\begin{aligned}(e \setminus x) \wedge (e \setminus y) &= (e \setminus x) \setminus ((e \setminus x) \setminus (e \setminus y)) \text{ by Defn. 1.4} \\ &= (e \setminus x) \setminus ((e \setminus (e \setminus y)) \setminus x) \text{ by S3} \\ &= (e \setminus x) \setminus (y \setminus x)\end{aligned}$$

But $(e\setminus x)\wedge(e\setminus y) = (e\setminus y)\setminus((e\setminus y)\setminus(e\setminus x))$ which, using the above procedures is equal to $(e\setminus y)\setminus(x\setminus y)$. Therefore, $y\wedge(e\setminus((e\setminus y)\setminus(x\setminus y))) = y\wedge(x\vee y) = y$.

<u>Thm. 1.23</u> $x \vee y$ is the least upper bound for x and y, if $x \vee y$ exists.

<u>Pf.</u> Suppose $z \in S$ is an upper bound for x and y. We need to show that $x \vee y \leq z$. That is, we need to show that $(x \vee y)\setminus z = o$.

$$\begin{aligned}(x \vee y)\setminus z &= (e\setminus((e\setminus x)\setminus(y\setminus x)))\setminus z \quad \text{by Defn. 1.6}\\ &= (e\setminus z)\setminus((e\ x)\setminus(y\ x)) \quad \text{by S3}\\ &= (e\setminus z)\setminus((e\setminus x)\wedge(e\setminus y)) \quad \text{by the note in the proof of the preceeding theorem.}\end{aligned}$$

One of the properties of upper bound is that $x, y \leq z$. This implies that $e\setminus x, e\setminus y \geq e\setminus z$. Therefore, $(e\setminus x)\wedge(e\setminus y) \geq e\setminus z$, which implies that $(e\setminus z)\setminus((e\setminus x)\wedge(e\setminus y)) = o$. Therefore, $x \vee y \leq z \ \forall z \in S \ni x, y \leq z$.

The following sets of theorems are used to demonstrate that $(e\setminus x)\setminus(y\setminus x) = (e\setminus x)\setminus y$, but are useful theorems in themselves.

<u>Thm. 1.24</u> $\forall\ x, y, z \in S,\ (z\setminus y)\setminus(z\setminus x) = (x\setminus y)\setminus(x\setminus z)$

<u>Pf.</u>
$$\begin{aligned}(z\setminus y)\setminus(z\setminus x) &= (z\setminus(z\setminus x))\setminus y \quad \text{by S3}\\ &= (x\setminus(x\setminus z))\setminus y \quad \text{by S2}\\ &= (x\setminus y)\setminus(x\setminus z) \quad \text{by S3}\end{aligned}$$

We will now use Thm. 1.24 to prove the following theorem.

<u>Thm. 1.25</u> $\forall\ x, y, z \in S \quad (z\setminus y)\setminus x = (z\setminus x)\setminus(y\setminus x)$

<u>Pf.</u> First, we will show that $(z\setminus y)\setminus x \leq (z\setminus x)\setminus(y\setminus x)$

$$\begin{aligned}((z\setminus y)\setminus x)\setminus((z\setminus x)\setminus(y\setminus x)) &= ((z\setminus x)\setminus y)\setminus((z\setminus x)\setminus(y\setminus x)) && \text{by S3}\\ &= ((z\setminus x)\setminus((z\setminus x)\setminus(y\setminus x)))\setminus y && \text{by S3}\\ &= ((y\setminus x)\setminus((y\setminus x)\setminus(z\setminus x)))\setminus y && \text{by S2}\\ &= ((y\setminus x)\setminus y)\setminus((y\setminus x)\setminus(z\setminus x)) && \text{by S3}\\ &= o\setminus((y\setminus x)\setminus(z\setminus x)) && \text{by Thm. 1.7}\\ &= o && \text{by Thm. 1.4}\end{aligned}$$

Now we will show that $(z\setminus x)\setminus(y\setminus x) \leq (z\setminus y)\setminus x$.

$$\begin{aligned}((z\setminus x)\setminus(y\setminus x))\setminus((z\setminus y)\setminus x) &= ((z\setminus x)\setminus((z\setminus x)\setminus y))\setminus(y\setminus x) && \text{by S3}\\ &= (((z\setminus x)\setminus x)\setminus((z\setminus x)\setminus y))\setminus(y\setminus x) && \text{by Thm. 1.1}\\ &= ((y\setminus x)\setminus(y\setminus(z\setminus x)))\setminus(y\setminus x) && \text{by S5}\\ &= o && \text{by Thm. 1.7}\end{aligned}$$

<u>Thm. 1.26</u> $\forall\ x, y, z \in S \quad z\setminus(z\setminus(y\setminus x)) = (z\setminus(z\setminus y))\setminus x$

<u>Pf.</u>
$$\begin{aligned}z\setminus(z\setminus(y\setminus x)) &= (y\setminus x)\setminus((y\setminus x)\setminus z) && \text{by S2}\\ &= (y\setminus x)\setminus((y\setminus z)\setminus x) && \text{by S3}\\ &= (y\setminus(y\setminus z))\setminus x && \text{by Thm. 1.25}\\ &= (z\setminus(z\setminus y))\setminus x && \text{by S2}\end{aligned}$$

<u>Thm. 1.27</u> (Isotone) $x \leq y \Rightarrow \ x\setminus z \leq y\setminus z$

Pf. $(x \setminus z) \setminus (y \setminus z) = (x \setminus y) \setminus z$ by Thm. 1.25
$= o \setminus z$ since $x \setminus y = o$
$= o.$

Therefore, $x \setminus z \leq y \setminus z$.

Thm. 1.28 If $x \vee y$ exists, $x, y \leq e$, then $x \vee y = e \setminus ((e \setminus x) \setminus y)$.

Pf. By Defn. 1.6, $x \vee y = e \setminus ((e \setminus x) \setminus (y \setminus x))$. By Thm. 1.25, $x \vee y = e \setminus ((e \setminus x) \setminus y)$.

The above definitions and theorems were presented so that we can prove theorems relating subtraction algebras to structures that we wish to study. This will be dealt with in the next chapter.

CHAPTER 2: Manuals

One structure which is extremely useful in the field of empirical science is the semi-Boolean algebra. Thus, in this chapter we will prove the equivalence of subtraction algebras and semi-Boolean algebras. After this is done, we will go on and define a manual, which is a special semi-Boolean algebra which Foulis and Randall first defined in their search for a mathematical structure. Finally, we will prove a few theorems regarding which semi-Boolean structures are manuals, and which are not.

Defn. 2.1 A Boolean algebra is a set B with a partial order, closed under greatest lower bound ($\wedge$) and least upper bound ($\vee$), with a smallest element (o) and a greatest element (1). Each element has its own relative complement with respect to 1 (known as its complement) which lies in set B. In addition, under $\vee$ and $\wedge$, set B must satisfy these associative, commutative, idempotent, absorptive, distributive, and DeMorgan laws.

Defn. 2.2 A meet semi-lattice is a set with a partial order such that meets are uniquely defined for every pair of elements.

Defn. 2.3 (a) An ideal I of a meet semi-lattice is a subset S such that if $x, y \in I$ then

1) if $z \leq x$, then $z \in I$

2) if $x, y \in I$ and if $x \vee y$ exists, then $x \vee y \in I$.

(b) A filter F is a subset of a meet semi-lattice such that if $x \in F$ and

1) if $x \leq z$, then $z \in F$

2) if $x, y \in F$, then $x \wedge y \in F$.

Defn. 2.4 A principal ideal I is an ideal in which $\exists\, x \in I$ such that $y \leq x$ for every $x \in I$.

Defn. 2.5 A semi-Boolean algebra is a meet semi-lattice in which every principal ideal is a Boolean algebra.[3]

Thm. 2.1 A subtraction algebra is a semi-Boolean algebra.

Pf. In the previous chapter we showed that the operations of "$\vee$" and "$\wedge$," defined in terms of a subtraction operation, are respectively the least upper bound and the greatest lower bound. We have already defined a partial order. Clearly there is a least element, $x \setminus x = o$, and in each principal ideal the greatest element exists by definition. Each element has its own complement, the relative complement with respect to the greatest element, as defined in the previous chapter. Furthermore, the commutative laws ($a \wedge b = b \wedge a$ and $a \vee b = b \vee a$), the associative laws ($a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$), and the idempotent laws ($x \vee x = x = x \wedge x$) are satisfied by the properties of the greatest lower bound and least upper bound. One of the DeMorgan laws follows directly from the definition of "$\vee$" found in the previous chapter: where e is least upper bound of principal ideal,

$$
\begin{aligned}
(x \vee y)^{\perp}_{e} = e\setminus(x \vee y) &= e\setminus(e\setminus((e\setminus x)\setminus(y\setminus x))) \\
&= e\wedge((e\setminus x)\setminus(y\setminus x)) \\
&= (e\setminus x)\setminus(y\setminus x) \\
&= (e\setminus x)\setminus((e\setminus(e\setminus y))\setminus x) \\
&= (e\setminus x)\setminus((e\setminus x)\setminus(e\setminus y)) \\
&= (e\setminus x)\wedge(e\setminus y) \\
&= x^{\perp}_{e}\wedge y^{\perp}_{e}.
\end{aligned}
$$

$$(x \wedge y)^{\perp}_{e} = (x^{\perp\perp}_{ee}\wedge y^{\perp\perp}_{ee})^{\perp}_{e} = (x^{\perp}_{e}\vee y^{\perp}_{e})^{\perp\perp}_{ee} = x^{\perp}_{e}\vee y^{\perp}_{e}$$

We will now prove the distributive laws.

We want to show that $x \vee (y \wedge z) = (x \vee y)\wedge(x \vee z)$. Let $r = (x \vee y)\wedge(x \vee z)$. Since $x \le x \vee y$, $x \vee z$, then $x \le (x \vee y)\wedge(x \vee z)$. Note also that $y \wedge z \le y \le x \vee y$ and $y \wedge z \le z \le x \vee z$. So, again $y \wedge z \le r$. Since x, $y \wedge z \le r$, $x \vee (y \wedge z) \le r = (x \vee y)\wedge(x \vee z)$. Now we need to show $x \vee (y \wedge z) \ge r$. Let $s = x \vee z$. $r \le s$, and so $r\setminus x \le s\setminus x$.

Since $(s\setminus x)\setminus z = o = (s\setminus z)\setminus x$, we have $s\setminus x = (s\setminus((s\setminus z)\setminus x)\setminus x =$ $(s\setminus(s\setminus z))\setminus x = z\setminus x \le z$. Therefore, $r\setminus x \le s\setminus x = z\setminus x \le z$. Similarly, we can show that $r\setminus x \le y$. Therefore, $r\setminus x \le y \wedge z$, and thus $(r\setminus x)\setminus(y\wedge z) = o$. $r = r\setminus o = r\setminus((r\setminus x)\setminus(y\wedge z)) =$ $x \vee (y \wedge z)$, by definition of "$\vee$".
Therefore $x \vee (y\wedge z) = (x\vee y)\wedge(x\vee z)$.

To show the other distributive property, we use the DeMorgan Laws:

$$
\begin{aligned}
x \wedge (y \vee z) &= x^{\perp\perp}\wedge(y \vee z)^{\perp\perp} = (x^{\perp}\vee (y \vee z)^{\perp})^{\perp} = (x^{\perp}\vee(y^{\perp}\wedge z^{\perp}))^{\perp} \\
&= ((x^{\perp}\vee y^{\perp})\wedge(x^{\perp}\vee z^{\perp}))^{\perp} = ((x \wedge y)^{\perp}\wedge(x\wedge z)^{\perp})^{\perp} \\
&= (x \wedge y)^{\perp\perp}\vee (x \wedge z)^{\perp\perp} \\
&= (x \wedge y)\vee(x \wedge z).
\end{aligned}
$$

The absorptive laws are simply consequences of the distributive laws.

Now that we have shown that a subtraction algebra is a semi-Boolean algebra, we would like to show that a semi-Boolean algebra is a subtraction algebra.

<u>Thm. 2.2</u> $x\setminus y = (x \wedge y)'_{x}$.

<u>Pf.</u> $(x \wedge y)'_{x} = x\setminus(x \wedge y) = x\setminus(x\setminus(x\setminus y))$, which is $x\setminus y$ by Defn. 1.5 and Thm. 1.13.

<u>Thm. 2.3</u> A semi-Boolean algebra is a subtraction algebra.

<u>Pf.</u> We will show S1, S2, and S3.

(S1) $x\setminus(y\setminus x) = (x \wedge (y\setminus x))^{\perp}_{x} = (x \wedge (y\setminus(x \wedge y))^{\perp}_{x}$
$= (x \wedge (x \wedge y)^{\perp}_{y})^{\perp}_{x} = ((x \wedge y)\wedge(x \wedge y)^{\perp}_{y})^{\perp}_{x} = o^{\perp}_{x} = x$

(S2) $x\setminus(x\setminus y) = (x \wedge (x \wedge y)^{\perp}_{x})^{\perp}_{x} = x^{\perp}_{x} \vee (x \wedge y) = o \vee (x \wedge y) = x \wedge y$.
Since $x \wedge y$ is commutative, $x\setminus(x\setminus y) = y\setminus(y\setminus x)$

(S3) $(x \setminus y) \setminus z = ((x \wedge y)'_x \wedge z)'_{x \setminus y}$ by Thm. 2.2. Note that $(x \wedge y)'_x = x \setminus (x \wedge y) = x \setminus y$. Also, $(x \setminus y) \wedge z \leq x \setminus y \leq x$. By the compatibility theorem 1.20 in the previous chapter,

$$((x \setminus y) \wedge z)^{\perp}_{x \setminus y} = ((x \setminus y) \wedge z)^{\perp}_x \wedge (x \setminus y)$$
$$= ((x \setminus y)^{\perp}_x \vee (z)^{\perp}_x) \wedge (x \setminus y)$$
$$= ((x \wedge y) \vee (x \wedge z)^{\perp}_x) \wedge (x \wedge y)^{\perp}_x$$
$$= ((x \wedge y) \wedge (x \wedge y)^{\perp}_x) \vee ((x \wedge z)^{\perp}_x \wedge (x \wedge y)^{\perp}_x)$$

by the distributive law

$$= o \vee ((x \wedge z)^{\perp}_x \wedge (x \wedge y)^{\perp}_x) = (x \wedge z)^{\perp}_x \wedge (x \wedge y)^{\perp}_x$$

Since this is symmetric in y and z, we have $(x \setminus y) \setminus z = (x \setminus z) \setminus y$.

What we have just demonstrated is that when we are talking about a semi-Boolean algebra, we can talk about a subtraction algebra, and vice-verse. This is useful in empirical logic, because the structures which the Foulis-Randall school uses are special types of semi-Boolean algebras. We can use some of the properties of subtraction algebras in dealing with these structures.

Now that the groundwork has been laid in subtraction and semi-Boolean algebras, we can begin developing the mathematics of empirical science. Let us, as usual, begin with some definitions.[4]

Defn. 2.6 A dominating set is a set M such that every x is contained in an e, $e \in M$ and if $\exists x$ in the algebra such that $e \leq x$ then $x = e$.

Defn. 2.7 An operational complement to an element x is an element y such that $x, y \leq e \in M$ and $e \setminus x = y$. An alternate definition is to say that $x \wedge y = o$ and $x \vee y = e \in M$. We write x oc y.

Defn. 2.8 An element x is said to be operationally perspective to another element if they are operational complements to the same element, w: that is, x oc w, w oc y, implies that x op y. We can also say that x op y iff e, $f \in M$, $x \leq e$, $y \leq f$, such that $e \setminus x = f \setminus y$.

Defn. 2.9 In the above definition, we call w the center of perspectivity for x and y.

Thm. 2.4 Let e, $f \in M$. Then $e \setminus f$ oc $e \wedge f$.

Pf. $e \setminus f \leq e$, $e \wedge f \leq e$. $e \setminus (e \setminus f) = e \wedge f$ by definition.

Cor. 2.4 (a) $e \setminus f$ op $f \setminus e$

Thm. 2.5 If $z \leq e \wedge f$, $\exists x$, such that $e \setminus f \leq x \leq e$ such that x oc z.

Pf. Let $x = e \setminus z$. Clearly $x = e \setminus z \leq e$. Since $z \leq e \wedge f \leq e$, $x = e \setminus z \geq e \setminus (e \wedge f) = e \setminus f$.

The above theorem states each element in the ideal of the meet of two dominating elements e and f is a center of perspectivity for elements in the filters of $e \setminus f$ and $f \setminus e$.

Thm. 2.6 If $e \geq x \geq e \setminus f$, then $\exists z \leq e \wedge f$ such that $e \setminus x = z$

Pf. $e \setminus x \leq e \wedge f$ iff $(e \setminus x) \setminus (e \wedge f) = o$. $(e \setminus x) \setminus (e \wedge f) = (e \setminus x) \setminus (e \setminus (e \setminus f))$

Since $x \geq e \setminus f$, then $e \setminus x \leq e \setminus (e \setminus f)$, which implies $(e \setminus x) \setminus (e \setminus x) \geq (e \setminus x) \setminus (e \setminus (e \setminus f))$. Since $o \geq (e \setminus x) \setminus (e \wedge f) \geq o$, $o = (e \setminus x) \setminus (e \wedge f)$ and $e \setminus x \leq e \wedge f$.

Cor. 2.6(a) There exists an element $y \geq f \setminus e$ such that x op y.

Pf. Thm. 2.5 and Thm. 2.6.

Thm. 2.7 Operational complements with respect to a single maximal element (dominating) are unique.

Pf. Assume $z_1, z_2 \leq e$, where $e \setminus z_1 = e \setminus z_2$. Then $e \setminus (e \setminus z_1) = e \setminus (e \setminus z_2)$, which implies that $z_1 = e \wedge z_1 = e \wedge z_2 = z_2$.

Cor. 2.7(a) For each element in the filter generated by the difference $e \setminus f$ of two maximal elements e and f, there is one and only one element oc to it in the ideal of $e \wedge f$ and one and only one element op to it in the filter of $f \setminus e$. Further, these are bijective relations.

Pf. Thms. 2.5, 2.6, and 2.7.

Thm. 2.8 If x oc y, $x, y \leq e \in M$, then $e \setminus x$ oc $e \setminus y$.

Pf. $e \setminus (e \setminus x) = e \wedge x = x = e \setminus y$.

Thm. 2.9 If x op y, then $x \setminus y$ op $y \setminus x$.

Pf. x op y implies that $\exists e, f \in M \ni x \leq e, y \leq f$ and $e \setminus x = f \setminus y$. This implies that $(e \setminus x) \vee (x \wedge y) = (f \setminus y) \vee (x \wedge y)$. We want to show that $x \setminus y$ oc $(e \setminus x) \vee (x \wedge y)$ oc $y \setminus x$. But $[(e \setminus x) \vee (x \wedge y)] \vee (x \setminus y) = (e \setminus x) \vee x = e$. Also $[(e \setminus x) \vee (x \wedge y)] \wedge (x \setminus y)] = [(e \setminus x) \wedge (x \setminus y)] \vee [(x \wedge y) \wedge (x \setminus y)] = [(e \setminus x) \vee (x \setminus y)] \vee o \leq (e \setminus x) \vee x = o$. Therefore, $[(e \setminus x) \vee (x \wedge y)]$ oc $x \setminus y$. Similarly for $y \setminus x$ oc $[(e \setminus x) \vee (x \wedge y)]$. So $x \setminus y$ op $y \setminus x$.

Thm. 2.10 If x oc y, $x, y \leq e \in M$, then $e \setminus x = y$.

Pf. If x oc y, $\exists f \in M \ni f \setminus x = y$. By definition, $x, y \leq f$. Let $e \setminus x = z$. $e \setminus z$ op $f \setminus y$ implies that $e \setminus (e \setminus z) = f \setminus (f \setminus y)$. Therefore, $e \wedge z = f \wedge y$ or $z = y$. So then $e \setminus x = f \setminus x$. $x \vee (e \setminus x) = x \vee (f \setminus x)$, and as a result, $e = f$.

The above theorem shows us that if two elements are oc and are contained in the same maximal element, then they are oc via that maximal element.

Defn. 2.10 A meet semi-lattice is dominated if $\exists$ a dominating set.

Defn. 2.11 A meet semi-lattice is atomic if $\exists$ a set A of atoms in the meet semi-lattice A such that $a \in A$, $x \in A$, $x \leq a$ implies that either $x = a$ or $x = o$, and every element, except o contains an atom.

Defn. 2.12 Condition M is satisfied if $\forall x, y, z \in A \ni x$ oc y, y op z, then x oc z. It is important in our study of empirical science for condition M to be satisfied, because we need this property to ensure that op is transitive. The need for this will become more apparent when we look at example problems.

Defn. 2.13 A dominated, atomic, semi-Boolean algebra satisfying condition M (DASBAM) is a manual.

Defn. 2.14 A test or operation is a maximal element in a manual.

Defn. 2.15 An atom or outcome is an element in the atomic set.

Defn. 2.16 An event is an element of the manual.

From Defn. 2.14, we can see where the term "operational complement" gets its name. It is derived from the fact that they are complements relative to an operation. "Operationally perspective" means that two events are the same, regardless of which operation gives you that interpretation.

We will now go on to describe different types of manuals which are of interest.

Defn. 2.17 Define a direct product on DASBA's as $\{(a,b) | a \in A, b \in B\}$, where A, B are DASBA's. Let the subtraction be defined as follows:

$$(x_1, x_2)\setminus(y_1,y_2) = (x_1\setminus y_1, x_2\setminus y_2),\ x_1, y_1 \in A,\ x_2, y_2 \in B.$$

Thm. 2.11 The direct product of two manuals is a manual.

Pf. The first requirement of a manual is that it be dominated. We propose that the dominating set be (e_1, e_2) such that $e_1 \in M_A$, $e_2 \in M_B$. Let $(x_1, x_2) \in A \times B$ such that $(e_1, e_2)\setminus(x_1, x_2) = (o,o)$, the "zero" of our new structure. This means that $e_1 \setminus x_1 = o$, $e_2\setminus x_2 = o$. Since A and B are dominated, $e_1\setminus x_1 = o$ implies that $e_1 = x_1$, and $e_2\setminus x_2 = o$ implies that $e_1 = x_2$. Therefore our dominating set consists of $M_A \times M_B$.

The next thing we need to show is that we have an atomic set. We propose that (o, a_2) and $(a_1\ o)$, $a_1 \in A_A$ and $a_2 \in A_B$ are then only atoms. Let $(x_1, x_2) \leq (o, a_2)$. Then $(x_1, x_2)\setminus(o, a_2) = (o, o)$, which means that $x_1\setminus o = o$ and $x_2\setminus a_2 = o$. This implies that $x_1 = o$ and $x_2 = a_2$, since $a, \in A_B$. And thus $(o, a_2) \in A_{A\times B}$ We can show similarly that $(a_1, o) \in A_{A\times B}$.

Next, we need to show that our structure is a semi-Boolean algebra. We will do this by varifying the axioms of a subtraction algebra. Let x_1, y_1, z_1 ε A, x_2, y_2, z_2 ε B.

S1) $(x_1,x_2)\setminus((y_1,y_2)\setminus(x_1,x_2)) = ((x_1\setminus(y_1\setminus x_1), x_2\setminus(y_2\setminus x_2)) = (x_1, x_2)$

S2) $(x_1,x_2)\setminus((x_1,x_2)\setminus(y_1,y_2))=(x_1\setminus(x_1\setminus y_1),x_2\setminus(x_2\setminus y_2))=(y_1\setminus(y_1\setminus x_1),$
$y_2\setminus(y_2\setminus x_2)) = (y_1,y_2)\setminus((y_1,y_2)\setminus(x_1,x_2))$

S3) $((z_1,z_2)\setminus(x_1,x_2))\setminus(y_1,y_2)=((z_1\setminus x_1)\setminus y_1,(z_2\setminus x_2)\setminus y_2)$
$= ((z_1\setminus y_1)\setminus x_1,(z_2\setminus y_2)\setminus x_2)$
$= ((z_1 z_2)\setminus(y_1,y_2))\setminus(x_1,x_2)$

Finally, we need to show that our new structure satisfies condition M. To do this, we need to explore first the definitions of oc and op in $A \times B$. Let x_1,y_1, z_1 ε A, x_2,y_2,z_2 ε B. $(x_1,x_2)\text{oc}(y_1,y_2)$ implies that $\exists(e_1,e_2)$ ε $M_{A\times B}$ such that $(e_1,e_2)\setminus(x_1,x_2)=(y_1,y_2)$. This is true iff $e_1\setminus x_1 = y_1$, and $e_2\setminus x_2 = y_2$, which in A and B means x_1 oc y_1 and x_2 oc y_2. Since op was defined as two events which share a common oc, we can also see that (x_1,x_2) op (y_1,y_2) iff x_1 op y_1 and x_2 op y_2. Thus, it is trivial to show that (x_1,x_2) oc (y_1,y_2), (y_1,y_2) op (z_1,z_2) implies that (x_1,x_2) op (z_1,z_2), since x_1 op z_1, x_2 op z_2 in A and B respectively.

<u>Defn. 2.18</u> Given a manual A with a principal ideal I(a), let A^g be the extension of A obtained by adding to A an element x_1 for each x ε I(a) under the following constraints:

1) if $x \leq y$, x, y ε I(a), then $x_1 \leq y_1$, x_1, y_1 ε A^g

2) if x ε I(a), then $x \leq x_1$

3) if $x \leq y$, x, y ε A, then $x \leq y$ in A^g

4) if x, y, z ε A^g, then $x \leq y$ and $y \leq z$ implies that $x \leq z$.

A^g is called a <u>ghost</u> of the principal ideal of a in A.

<u>Defn. 2.19</u> <u>Dacification</u> is ghosting every maximal principal ideal. That is to say, we ghost each test.[5]

<u>Defn. 2.20</u> A <u>Dacey manual</u> is a manual in which every maximal element e contains an atom a_e such that if e, f ε M, e $\neq$ f, then $a_e \neq a_f$, where a_e, a_f ε A, $a_e \leq e$, and $a_f \leq f$.

<u>Thm. 2.12</u> A dacification of a dominated, atomic, semi-Boolean algebra (DASBA) is a manual.

Pf. Ghosting tests leaves us with a new dominated, atomic, semi-Boolean algebra. We must show that the dacification satisfies condition M. In the dacification, let x_1 oc w_1 oc y_1. We will use the subscript D to signify the dacified DASBA, and the subscript A to signify the original DASBA. $x_1 \leq e_1 \varepsilon M_D$, $y_1 \leq f_1 \varepsilon M_D$. $e_1 = e \vee a_e$, $e \varepsilon M_a$, $a_e \varepsilon A_D$. Similarly, we define f_1. w_1 cannot equal $w \vee a_e$ or $w \vee a_f$, $w \varepsilon A$, since $a_e \neq a_f$ and $w_1 \leq e_1, f_1$. Therefore, $w_1 = w$. That is, w_1 was in the original DASBA. Thus, $x_1 = x \vee a_e$, $y_1 = y \vee a_f$, $x, y \varepsilon A$. Is y_1 an operational complement to anything but w_1? The answer is no, since a_f is contained in one and only one test, f_1. Therefore, condition M is trivially satisfied, since w_1 is the only event oc to y_1, and we already have x_1 oc w_1.

Defn. 2.21 A classical manual is a manual in which there is only one element in the dominating set.

A classical manual is simply a Boolean algebra.

Defn. 2.22 A semi-classical DASBA is one in which the intersection of principal ideals of any two events in the dominating set is 0.

In the lattice drawing, a semi-classical DASBA appears to be two or more Boolean algebras which meet at 0.

Thm. 2.13 A semi-classical DASBA is a manual.

Pf. We just need to show the DASBA satisfies condition M. Note that the only op pairs in the structure are tests, since 0 is the only element in $e \wedge f$, $e, f \varepsilon M$. Therefore, proving condition M is rather trivial, since we only need to look at something of the following form: 0 oc e, e op f, which clearly satisfies condition M because 0 oc f.

Thm. 2.14 Any DASBA with one or two tests is a manual.

Pf. With one test, there are no op pairs, so condition M is vacuously satisfied. With two tests, if we have x oc y and y op z, then there exists $e, f \varepsilon M$ such that $e \setminus y = f \setminus z = w$, where $w \leq e, f$. $y \leq e$, and therefore $y \not\leq f$ since if it were, y would then be in the filter of $e \setminus f$, and the ideal of $e \wedge f$. Thus, $e \setminus y = x = w$, by Theorem 2.7.

Theorems 2.12, 2.13, and 2.14 simply give us an easy way to recognize some DASBAs as manuals.

CHAPTER 3: Logics

This chapter is one which discusses "op logics," or "logics" for short. Logics exhibit some properties which are useful in empirical science. For example, in a logic, equivalent events from the manual are identified. That is to say, if two events are operationally perspective, they share the same operational complement. So if x oc y, and w occurs (that is, when the test is performed we are assured of getting w), then x and y non-occur simultaneously. In this manner, x and y are equivalent, and in fact we will show that op is an equivalence relation in a manual. We will also demonstrate some properties exhibited by a logic.

<u>Thm. 3.1</u> "op" is an equivalence relation in a manual.

<u>Pf.</u>
(i) x op x, since x oc e\x for any $e \in M$ such that $x \leq e$.

(ii) x op y implies that y op x by symmetry of Definition 2.8.

(iii) Let x op y and y op z. y op z implies that there exists a w such that y oc w oc z. Since we are in a manual, condition M says that x oc w. Since w oc z, x op z.

<u>Defn. 3.1</u> Let $\bar{x}$ denote the set of events in the manual such that $y \in \bar{x}$ iff y op x.

Thus, $\bar{x}$ is the equivalence class of x and x op y iff $\bar{x} = \bar{y}$.

<u>Defn. 3.2</u> $\bar{x} \leq \bar{y}$ iff for every $x_1 \in \bar{x}$ there exists $y_1 \in \bar{y}$ such that $x_1 \leq y_1$.

<u>Thm. 3.2</u> $x_1, x_2 \leq y$ and x_1 op x_2 implies that $x_1 = x_2$.

<u>Pf.</u> If $x_1\ x_2 \leq y$, there exists an $e \in M$ such that $x_1 \leq y \leq e$. x_1 oc $e \setminus x_1$ and x_1 op x_2 implies that $e \setminus x_1$ oc x_2. This implies that there exists an $f \in M$ such that $x_2 \leq f$ and $e \setminus x_1 = f \setminus x_2$. $x_2 \vee (e \setminus x_1) = f$ and $x_2 \wedge (e \setminus x_1) = 0$. $x_2 \leq e$ and $(e \setminus x_1) \leq e$. Therefore $f \leq e$. In a manual, one test cannot be contained in another, so $f = e$. Therefore $e \setminus x_1 = e \setminus x_2$, which implies that $x_1 = x_2$.

This theorem tells us that two events in $\bar{x}$ do not share the same y in $\bar{y}$, $\bar{x} \leq \bar{y}$, such that y is greater than both of them.

<u>Thm. 3.3</u> " $\leq$ " is a partial order in the logic.

<u>Pf.</u>
(i) $\bar{x} \leq \bar{x}$, since $x \in \bar{x}$ and $x \leq x$.

(ii) $\bar{x} \leq \bar{y}$ and $\bar{y} \leq \bar{x}$ says that for every $x_1 \in \bar{x}$ there exists $y_1 \in \bar{y}$ such that $x_1 \leq y_1$, and for every $y_2 \in \bar{y}$ there exists an $x_2 \in \bar{x}$ such that $y_2 \leq x_2$. In the manual, $x_1 \leq y_1$. Also, $y_1 \leq x_2 \in \bar{x}$. There exists an $e \in M$ such that $x_1 \leq y_1 \leq x_2 \leq e$. But x_1 op x_2. By Theorem 3.2, $x_1 = x_2$. Therefore $\bar{x} = \bar{y}$.

(iii) Let $\bar{x} \leq \bar{y}$ and $\bar{y} \leq \bar{z}$. Then for every $x_1 \in \bar{x}$ there exists $y_1 \in \bar{y}$ such that $x_1 \leq y_1$. For this $y_1 \in \bar{y}$, there exists $z_1 \in \bar{z}$ such that $y_1 \leq z_1$. Therefore, $x_1 \leq z_1$ for every $x_1 \in \bar{x}$.

Therefore, we have established that the logic is a partially ordered set. We will use Theorem 3.4 to prove Theorem 3.5.

Thm. 3.4 Let $x \vee z$ op $y \vee z$ and $x \wedge z = y \wedge z = 0$. Then x op y.

Pf. $x \vee z$ op $y \vee z$ implies that there exists $e, f \in M$ such that $x \vee z \leq e$, $y \vee z \leq f$ and $w = e \setminus (x \vee z) = f \setminus (y \vee z)$. $w, z \leq e, f$, which implies that $w, z \leq e \wedge f$, which in turn shows us that $w \vee z \leq e \wedge f$. $e = w \vee (x \vee z) = (w \vee z) \vee x$ and $f = w \vee (y \vee z) = (w \vee z) \vee y$. Note that $(x \vee z) \wedge w = 0 = (y \vee z) \wedge w$. By the distributive law, this says that $(w \wedge x) \vee (w \wedge z) = 0 = (w \wedge y) \vee (w \wedge z)$, which says that $w \wedge x = w \wedge y = w \wedge z = 0$. $(w \vee z) \wedge x = (w \wedge x) \vee (z \wedge x) = 0 \vee 0 = 0$. Thus, x oc $w \vee z$ oc y, and as a result, x op y.

Thm. 3.5 x op y, $u \leq x$, $v \leq y$, and u op v implies $x \setminus u$ op $y \setminus v$.

Pf. x op y implies that there exists an $e \in M$, $x \leq e$, and an $f \in M$, $y \leq f$ such that $e \setminus x = f \setminus y = w$, where $e = x \vee w$ and $f = y \vee w$.

$x = u \vee (x \setminus u)$, where $u \wedge (x \setminus u) = 0$. $e = u \vee (x \setminus u) \vee w$, where $u \wedge w = 0$ and $(x \setminus u) \wedge w = 0$, since $x \wedge w = 0$. Similarly, $f = v \vee (y \setminus v) \vee w$. Therefore, u oc $(x \setminus u) \vee w$. Since u op v, and we are in a manual, v oc $(x \setminus u) \vee w$.

Therefore $(x \setminus u) \vee w$ op $(y \setminus v) \vee w$. By Thm. 3.4, $x \setminus u$ op $y \setminus v$.

Defn. 3.4 If $\bar{x} \leq \bar{y}$, then we define $\bar{y} \setminus \bar{x} = \overline{y \setminus x}$.

The reader will note that this definition was motivated by Thm. 3.5 which says that if $\bar{x} \leq \bar{y}$, $x_1, x_2 \in \bar{x}$, $y_1, y_2 \in \bar{y}$, $x_1 \leq y_1$ and $x_2 \leq y_2$, then $y_1 \setminus x_1$ op $y_2 \setminus x_2$. If $e \in M$, we will denote $\bar{e}$ by writing 1. This makes sense, because all of the tests are in the same op class, since each is an operational complement to 0. The equivalence class of 0, $\bar{0}$, consists only of the null event, 0.

Defn. 3.5 The orthocomplement of $\bar{x}$ is $1 \setminus \bar{x}$. We denote this as $\bar{x}^{\perp}$.

This makes good sense, as it is motivated by Definition 3.4.

Thm. 3.6 $\bar{x}^{\perp\perp} = \bar{x}$

Pf. $\bar{x}^{\perp\perp} = (1 \setminus \bar{x})^{\perp}$. $1 \setminus \bar{x} = \overline{e \setminus x}$ for some $e \in M$ such that $x \leq e$. $\overline{e \setminus x}^{\perp} = 1 \setminus \overline{e \setminus x}$. Since $e \setminus x \leq e$, $1 \setminus \overline{e \setminus x} = \overline{e \setminus (e \setminus x)} = \bar{x}$.

We have shown that the op logic has a least element and a greatest element, the logic has a partial order defined on it, and it also has a subtraction operation defined for one element contained in another. Finally, each element has a unique orthocomplement.

In any case, the main point that should be made about op logics is that in the logic, events which are essentially identical in physical interpretation, but different in the way in which the outcomes are recorded in the manual, are identified. This is appealing to one's intuition, since it seems unreasonable for identical occurances in the physical domain to be recorded as independent events.

Logics are partially ordered sets, but they do not always form lattices, and even when they do form lattices, they are not necessarily distributive. Later in Chapter 7, we will see some examples of logics.

CHAPTER 4: Notation and Computers

Up until this point, we have discussed manuals as semi-Boolean algebras. We have not really driven home the point that in the applicaton of the theory, maximal elements in the manual really represent physical experiments. Since this is the case, Stone's Representation Theorem, which shows that a Boolean algebra is equivalent to an algebra of sets, tells us that a test can be expressed as the set of its allowable outcomes. In this sense, then, we can talk of one event being less than another, or we can say that one event is contained in another.

Later in the chapter, we discuss a computer program which benefits from this clarification of interpretations. The program itself is extremely useful, in that it allows us to decide when a dominated, atomic, semi-Boolean algebra is a manual. Previous to this program, all verification was done by hand.

Defn. 4.1 $A_x = \{a \mid a \in A, a \leq x\}$ where x is an event and A is the set of atoms in the manual. A_x is actually "the set of atoms contained in x."

Defn. 4.2 $M_x = \{e \mid e \in M, x \leq e\}$ where x is an event and M is the set of maximal elements in the manual. M_x is actually "the set of tests containing x."

Defn. 4.3 $D_x = \{a \mid a \leq e, e \in M_x, a \in A\}$ where x is an event. D_x is spoken as the "domain of x."

Thm. 4.1 $x \leq y$ implies that $M_y \subseteq M_x$.

Pf. $M = \{e \mid e \in M, y \leq e\}$. But since $x \leq y$, then $x \leq e$ for every e such that $y \leq e$. Hence, $M_y \subseteq M_x$.

Thm. 4.2 $x \leq y$ implies that $A_x \subseteq A_y$.

Pf. $A_x = \{a \mid a \in A, a \leq x\}$. But for every a such that $a \leq x$, $a \leq y$. Therefore, $A_x \subseteq A_y$.

Thm. 4.3 $x \leq y$ implies that $D_y \subseteq D_x$.

Pf. This result follows directly from Theorems 4.1 and 4.2.

Thm. 4.4 If $a \leq x \setminus y$, $a \neq 0$, then $a \leq x$ and $a \nleq y$.

Pf. Let $a \leq x \setminus y \leq x$. Assume $a \leq y$. Then $a \leq x \wedge y$. Therefore, $a \leq (x \setminus y) \wedge (x \wedge y) = 0$. This is a contradiction, so $a \nleq y$.

Thm. 4.5 (Stone's Representation Theorem) $A_{x \setminus y} = A_x \setminus A_y$.

Pf. $a \in A_{x \setminus y}$ implies that $a \leq x \setminus y$, which in turn implies that $a \leq x$ and $a \nleq y$, by Thm. 4.4. Therefore, $a \in A_x$ and $a \in A \setminus A_y$ ($a \notin A_y$). So $a \in A_x \cap (A \setminus A_y) = A_x \setminus A_y$.

Cor. 4.5A $A_{x \vee y} = A_x \cup A_y$.

Cor. 4.5B $A_{x \wedge y} = A_x \cap A_y$.

These follow from the fact that a subtraction algebra defines a semi-Boolean algebra.

Thm. 4.6 $M_x \setminus M_y \subseteq M_{x \setminus y}$.

Pf. Let $e \in M_x \setminus M_y$. That is, $x \leq e$, $y \not\leq e$. $x \setminus y \leq x \leq e$, and therefore $e \in M_{x \setminus y}$.

Thm. 4.7 $M_{x \vee y} = M_x \cap M_y$.

Pf. Let $e \in M_{x \vee y}$. Then $x \vee y \leq e$, which implies that $x \leq e$ and $y \leq e$. Therefore, since $e \in M_x$ and $e \in M_y$, $e \in M_x \cap M_y$.

Let $e \in M_x \cup M_y$. Then $e \in M_x$ and $e \in M_y$. This says that $x \leq e$ and $y \leq e$. This in turn implies that $x \vee y \leq e$, and therefore $e \in M_x \vee M_y$.

Thm. 4.8 $M_x \cup M_y \subseteq M_{x \wedge y}$.

Let $e \in M_x \cup M_y$. Either $x \leq e$ or $y \leq e$. Without loss of generality, we can assume that $x \leq e$. Since $x \wedge y \leq x \leq e$, then $e \in M_{x \wedge y}$.

Cor. 4.8A $D_x \setminus D_y \subseteq D_{x \setminus y}$.

Cor. 4.8B $D_{x \vee y} = D_x \cap D_y$.

Cor. 4.8C $D_x \cup D_y \subseteq D_{x \wedge y}$.

Note that the domains follow the exact same rules as maximals by Stone's Representation Theorem.

Thm. 4.9 $x \vee y$ exists iff $M_x \cap M_y \neq \emptyset$.

Pf. Let $x \vee y$ exist. There exists an $e \in M$ such that $x \vee y \leq e$. $x, y \leq x \vee y \leq e$, and as a result $e \in M_x \; M_y$. Let $M_x \cap M_y \neq \emptyset$. This implies that there exists an $e \in M$ such that $e \in M_x \cap M_y$. Therefore, $e \in M_x, M_y$. $x, y \leq e$ then implies that $x \vee y \leq e$, since we have a Boolean algebra.

Defn. 4.4 $A_y^{\perp} = \{a \mid a \in A,\ a \wedge y = 0,\ a \vee y \leq e,\ e \in M\}$.

Thm. 4.10 $A_y^{\perp} = D_y \setminus A_y$.

Pf. Let $a \in A_y^{\perp}$. We need to show that $a \in D_y$ and $a \notin A_y$. But $a \wedge y = 0$, which means that $a \not\leq y$, or $a \notin A_y$. Furthermore, $a \vee y \leq e$ for some $e \in M$. Since $e \in M_y$ and $a \leq e$, then $a \in D_y$.

Let $a \in D_y \setminus A_y$. Then $a \in D_y$ and $a \in A_y$. Hence there exists an $e \in M$ such that $a \leq e$. Therefore, $a, y \leq e$, and $a \vee y \leq e$.

Since $a \notin A_y$, $a \wedge y = 0$. Thus, $a \in A_y$.

Now that we have completed the development of the notation, we will discuss what the program, MANUAL1 does.

The computer program, MANUAL1, was written in order that we might have a quick, easy method of verifying that a dominated, atomic, semi-Boolean algebra is a manual. Chapter 2 suggested that in some cases we could determine this very easily if it met one of several qualifications. For example, we will have a manual if the DASBA has only one or two tests, or if it is a dacification of another DASBA. However, in a majority of cases, it will not be readily apparent if the DASBA is a manual. And if we have a DASBA with more than a few tests or atoms, checking all possible combinations x oc y and y op z can be tedious, if not nearly impossible. Fortunately, the computer can be taught to do these things very nicely.

A listing of MANUAL1 is found in the Appendix, along with several sample runs. It was written in Fortran, though any computer language which had Boolean operations could have been used. To operate the program, we input the outcomes of one test into the computer, using alphanumeric symbols. The program itself can only handle the letters A to Z and the numbers 0 to 9; though, with modifications could handle any character. The program is also limited to 36 bits, though, with a little ingenuity, one could extend this limit.

The alphanumeric representation is immediately converted to a 36 digit binary representation, and in this representation we do all of the real work of this program. Since all decimal numbers are stored in binary, we can take advantage of this feature in generating the power set of the set of outcomes contained in a test. For example, if there are n atoms in a test, we take the binary form of the integers 0 to 2^{n-1}, and allow the corresponding event to contain the atoms that are in the same position as the 1's. Order and position are extremely important at this stage of the program.

The next thing that the program does is to generate the operational complement of each event with respect to the test. It does this by using the "exclusive or", the negation of the biconditional, with the test and the event. At this point, the events have been stored in one table, and the respective oc in a corresponding position in another table.

Once each test has been input and the two tables have been generated, the computer searches the oc table for an event which is listed twice. Their corresponding events are an op pair, and are listed in corresponding positions in two separate tables.

After this is completed, the program begins to check condition M. It does this by searching several tables at once. First it checks for an event in the oc table equal to an event in one of the op tables. Once we have a match, it searches the event table for all events identical to the event in the event column corresponding to the matched event in the oc column. It does this to check to see if the corresponding event in the event column is oc to the event in the second of the op tables corresponding to the matched op event. If it is oc, we continue the process. If it is not, we store the counter-example in three tables, and continue the search to find more counter-examples.

The program finally tells us whether or not we have a manual, and gives us sevral print out options, including an event/oc table and an op pair table. The program will also compute the "perp" of each event, and check the "coherence" condition, but these features are not relevant to this paper, and will not be discussed here.

Before printing is done, the binary representation is converted to the alphanumeric representation. Since each position of the 36-place conversion key has a corresponding numerical value equal to 2^{36-k}, where k is the position, we simply use numerical comparison and arithmetic subtraction to find the desired event in the form which we would recognize.

CHAPTER 5: Weights and States

In the previous four chapters, we have dealt with algebras, manuals, and logics. One thing that we have yet to discuss is some type of probability distribution on the manuals and logics. This is one area in empirical science which has been developed by many of the leaders in the field.

It is logical to discuss probabilities, since in classical manuals, one is accustomed to discussing the likelihood of various outcomes to a single experiment. Why, then, should we not discuss these probabilities, or "weights", on more than one experiment? There is no reason, and that is exactly what we will attempt to do.

<u>Defn. 5.1</u> A <u>weight</u> is a real value w, $0 \leq w \leq 1$, assigned to an event in the manual.

<u>Defn. 5.2</u> If n is the number of atoms in a manual, a <u>state</u> is an n-tuple assigning weights to all n atoms under the following conditions:

1) The weight of each event is equal to the sum of the weights of the atoms contained in it
2) The weight of each test is 1

In a very real sense, a weight is equivalent to what we would think it to be in terms of classical probability theory, formalized originally by Komolgorov. A state is simply a consistent way of assigning weights so that the sum of the weights assigned to the outcomes of a given experiment equals 1. This is because if the experiment were performed alone, classical theory would not permit this value to exceed 1, but would allow it only to equal 1.

<u>Defn. 5.3</u> A <u>pure state</u> is a state which cannot be expressed as a convex linear combination of any of the other states. That is to say, it is an extreme point of the convex hull of allowable solutions.

If we can find every pure state, then we can express all states as convex combinations of pure states. Thus, it would be good to find these pure states, so that we will know all about allowable states.

From the constraints which are set down for a state in Defintion 5.2, we can find extreme point solutions by using techniques borrowed from the simplex method in linear programming. The rest of the development in this chapter is taken form Gass's <u>Linear Programming</u>, third edition.[6] The theorems will be stated without proof, with only their uses being cited.

<u>Defn. 5.4</u> A <u>feasible solution</u> to the linear programming problem is an n-tuple in which each of the components is non-negative, and is a solution to a linear system of m equations and n unknowns.

In a manual with r tests and n atoms, we have r equations with the constraints that the sum of the atoms in each test is equal to 1. In addition, we have the constraint that the weight of each of the n atoms lie between 0 and 1. However, we should note that one of the conditions of a feasible solution is that each component is non-negative. In addition, each set of atoms in a test sums to 1. Therefore, we do not need the second set of n constraints, since $0 \leq w \leq 1$ has been satisfied by the first r equations. Therefore, r = m, and our system of m equations and n unknowns consists of summing the atoms in each test, so that they are equal to 1.

Defn. 5.5 A basic solution to a system of linear constraints as described in the previous definition is obtained by setting n-m variables equal to zero, and solving the remaining m x m augmented matrix. This assumes that the determinant is non-zero.

Thm. 5.1 The set of all feasible solutions to the linear programming problem is a convex set.

Thm. 5.2 If a set of $k \leq m$ vectors $P_1, \ldots, P_k$ can be found that is linearly independent and such that $x_1P_1 + \ldots + x_kP_k = P_0$ and all $x_i \geq 0$, then the point $X = (x_1, \ldots, x_k, 0, \ldots, 0)$ is an extreme point of the convex set of feasible solutions. Here X is an n-dimensional vector whose last n-k elements are 0.

Thm. 5.3 Associated with every extreme point of the convex hull of solutions is a set of m linearly independent vectors from the given set $P_1, \ldots, P_n$.

Thm. 5.4 $X = (x_1, \ldots, x_n)$ is an extreme point of the convex hull iff the positive x_j are coefficients of linearly independent vectors P_j in $\sum_{j=1}^{n} x_jP_j = P_0$.

Gass summarizes these pertinent theorems by saying that

1) Every basic, feasible solution corresponds to an extreme point of the convex hull
2) Every extreme point of the convex hull of solutions has m linearly independent vectors of the given set of n associated with it.

When we get to the chapter of examples, we will demonstrate how to use these results in a practical problem, and even suggest how a computer might be utilized in finding each of the pure states.

CHAPTER 6: Properties, States, and Questions

After studying logics, manuals, and weight, it became apparent that we were lacking a means of insight into these systems beyond what we had already attained, or these systems were simply building blocks — means of interpretation — that were pointing toward some better means of interpretation. Event though the logics and manuals told us much about the interaction of tests, it seemed that there should be more: though what that more should have been was difficult to describe. Nevertheless, in the search for understanding, one simply desired to know more than we did.

At that point, at the recommendation of Foulis and Randall, we took a look at a doctoral thesis by Aerts entitled The One and the Many.[7] Aerts was a student of Constantin Piron, director of the Institute of Theoretical Physics in Geneva, who was working with Foulis and Randall at the time, each of them trying to come to terms with the other's work.[8] It seems that they more or less have been doing the same things, though similar terms had different meanings. What our understanding of Piron' and Aerts' work in relation to the work of Foulis and Randall is found in this chapter.

Defn. 6.1 An entity is an object or system on which tests may be performed.

Defn. 6.2 A question is a means of making a test, the result of which can be interpreted either "yes" or "no."

Defn. 6.3 Question α is said to be stonger than question β ($\alpha \prec \beta$) iff whenever α is "true", β is "true."

One will find that " " exhibits the properties of a preorder. That is, (i) $\alpha \prec \alpha$ and (ii) if $\alpha \prec \beta$ and $\beta \prec \gamma$, then $\alpha \prec \gamma$. One can verify this simply by following through the definition of "$\prec$."

Defn. 6.4 Question α is equivalent to question β ($\alpha = \beta$) iff $\alpha \prec \beta$ and $\alpha \succ \beta$.

As a result of this definition, "$\prec$" is a partial order. That is, in addition to the two conditions satisfied as a preorder, it also now satisfies that $\alpha \prec \beta$, $\beta \prec \alpha$ implies that $\alpha = \beta$.

Defn. 6.5 A property is an equivalence class of questions.

Defn. 6.6 A property a is actual if there exists a question $\alpha \varepsilon a$ such that α is true. If not, a is said to be potential.

Defn. 6.7 An Aerts' state is the collection of actual properties of an entity.

Usually we will call an Aerts' state simply a state. This is not to be confused with the term "state" used in the previous chapter. Usually, it will be clear from context which state that we are talking about.

Defn. 6.8 A product of questions is a set of two or more questions of which only one is chosen to be tested. This is written as $\alpha \cdot \beta$ or $\Pi\alpha_i$, where α, β, and α_i are questions.

In other words, for a product of questions to be true, whichever question is chosen to be tested must be true. This does not mean that we must test each question; rather, if we did test any one of them, it would be true.

Defn. 6.9 A primitive question is a question which can be tested by one experimental set up.

Defn. 6.10 A product of properties, $a \cdot b$, is two properties which can be tested by a product of questions, $\alpha \cdot \beta$, where $\alpha \varepsilon a$, $\beta \varepsilon b$.

Defn. 6.11 A primitive property a is one such that there exists a primitive question $\alpha \varepsilon a$.

Now that some of the terms have been defined, we will define a partial order on the properties, state some results from Aerts' development, and then discuss how the Aerts development relates to the Foulis-Randall development.

Defn. 6.12 Let a and b be properties. $a \prec b$ iff $\alpha \prec \beta$, where $\alpha \varepsilon a$, $\beta \varepsilon b$.

Defn. 6.13 $a = b$ iff $a \prec b$ and $b \prec a$.

"$\prec$" is a partial order on the properties. Aerts shows later that the properties form a complete lattice, with the greatest lower bound of two properties being represented by the product of two properties, and the least upper bound being defined as in Defn. 6.14.

Defn. 6.14 Let a and b be properties. The least upper bound of a and b (a   b) satisfies the following requirements:

1) $a, b \prec a \quad b$
2) $a, b \prec c$, where c is a property, implies that $a \vee b \prec c$.

Aerts' primitive properties apparently come from the event lattice. Clearly, events correspond to primitive properties, since each event can be tested by one experimental set up. Are these the only primitive properties? The answer is "yes", since a manual defines all the basic responses that are allowed on an entity.

Foulis and Randall have asserted that a property lattice is an inverted manual filter lattice. Before we discuss this further, it would be wise to first define a manual filter.

Defn. 6.15 F is a manual filter on a manual if F is non-empty and for a and b in the manual we have the following:

1) if $a \varepsilon F$, $b \prec a$, then $b \varepsilon F$
2) if $a \varepsilon F$, a op b, then $b \varepsilon F$
3) if a, $b \varepsilon F$ and $a \vee b$ exists, then $a \wedge b \varepsilon F$.

The assertion is reasonable if one considers the definition of the manual filter. If event a is assigned a weight of 1, it would be reasonable to expect that any event greater than a or operationally perspective to a would also be assigned a weight of 1. In addition, if both a and b are assigned weights of 1, $a \nleq b$ and $b \nleq a$, and both are in the same test, then the atoms which are contained in a are summed to 1 and the atoms contained in b are summed to 1. Unless the atoms in $a \wedge b$ sum to 1, we have a case where the atoms in a single test sum to a number greater than 1, a condition which is not allowed. Since a weight of 1 corresponds to an affirmative response to a question, it is in this way that a filter set ordering corresponds to a property lattice ordering.

However, the ordering is inverted since if $a \leq b$, a, b events, the manual filter of b is contained in the manual filter of a, but if a is true, then b is true. So the Foulis-Randall assertion indeed seems reasonable.

In some of the examples in Chapter 7, we will look at filters and properties a little more closely.

CHAPTER 7: Examples

This chapter is intended to give the reader a better understanding of the topics discussed in previous chapters by illustrating some of those concepts. The material in this chapter is presented in such a way that one could get a "feel" for what empirical science is attempting to accomplish, but for a more rigorous approach the reader is referred to the previous six chapters.

This paper attempts to strike a balance between the theory of empirical science and the application of this field to the real world. This balance is important for several reasons. Naturally, without mathematical rigor, our empirical techniques are worthless. And without applications, empirical science is just another concept which exists only in man's minds. Fortunately, it seems that many things that one encounters every day can be better understood through empirical science, though, as was mentioned before, empirical science has been sought to explain things as uncommon to most of us as quantum physics.

Empirical science has actually evolved from classical probability theory which was essentially formalized by Komolgorov in the 1930's. At the time, Komolgorov was dealing with outcomes of one test or experiment which was isolated from the rest of the world. If we wanted to test more than one experiment at the same time, especially if the outcome of one effected the outcome of another, then we would have a completely different situation.

In quantum physics, we might ask, "What is the position?" of a given particle in a physical system, and then assign a probability function to the response, but what if we ask at the same time, "What is the momentum?" The Heisenberg Uncertainty Principle tells us that asking the second question will effect the outcome of the first question. So here we have an example of what ultimately empirical science is trying to accomplish: to be able to consistently explain the relationship between outcomes of more than one test. Komolgorov's classical probability theory is unable to deal with circumstances such as this.

Empirical science should be useful in many facets of life. For example, public opinion polls seem to be quite popular presently. But the real challenge is to be able to correctly interpret the outcome to these polls. And the reason that this is a challenge is that a poll is really a series of tests, generally each having just a few outcomes. Understanding the relationship between these tests can significantly effect the interpretation. For example, most people would respond that they are in favor of religious freedom in the United States, but if they were first asked if cults, like the one that lead to the Jonestown massacre, should be prohibited, people might respond differently. The point being, the Jonestown question would cause people to respond differntly to the religious freedom question.

What if in an election poll people selected candidate A over candidate B two to one, candidate B over candidate C two to one, and candidate C over candidate A two to one? How would we interpret this? Empirical science strives to understand things like this.

In the Foulis-Randall school, we are concerned with manuals (see Chapter 2). A manual is essentially a list of instructions as to what tests (or operations or experiments) are to be performed, and which are the allowable outcomes (or atoms). A logic is simply an attempt to tie together elements of the manual (events) which are equivalent in a very real sense, without regard for the tests in which the events are contained.

Let us now begin by looking at some examples.

Example 1. We will begin with a very basic example, which we will build on later to make a more complicated example. Once the reader examines our mathematical structures, he will find out that a number of different scenarios can be fit into them. For this example, let us assume that we have a port lookout aboard a ship, who can make the following reports as to whether he has sighted a specific buoy which we desire to locate:

- 1 if the buoy is located within 2 points (22.5 degrees) to either side of dead ahead
- 2 if he does not see the buoy
- 3 if the buoy is located within 2 points off the port bow and dead astern (180 degrees to 337.5 degrees relative)
- 12 if the buoy is definitely not in sector 3
- 13 if the buoy has been sighted by the port lookout
- 23 if the buoy is definitely not in sector 1
- 123 if the lookout is looking for the buoy
- 0 if the lookout is not on watch

For simplicity's sake, we will call sector 1 "dead ahead", sector 3 "port quarter", and sector 13 "port side." Note that the port lookout cannot see the starboard quarter, perhaps due to the blockage of the superstructure. A drawing of the situation appears in Figure 1.1.

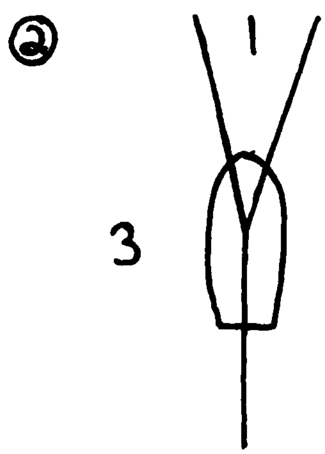


Figure 1.1. Schematic drawing of this example.

We will represent this manual consisting of one test (123) first by drawing a Greechie diagram. Recall that a manual with one test is called a classical manual, since the sum of the probabilities on the allowable outcomes is 1.

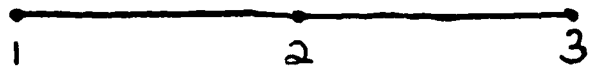


Figure 1.2. Greechie diagram of the classical manual in example 1.

A Greechie diagram is a simple way of representing a manual. Each test is represented by a straight line, and each outcome contained in that test is represented by a point on that line. A Greechie diagram is simply a shorthand way of describing a manual. It is quick and easy to draw and use.

However, it goes a long way from describing some more important aspects of a manual. A better way is to draw this classical manual as a semi-Boolean algebra, in which case we get the following, more informative structure, called the event lattice.

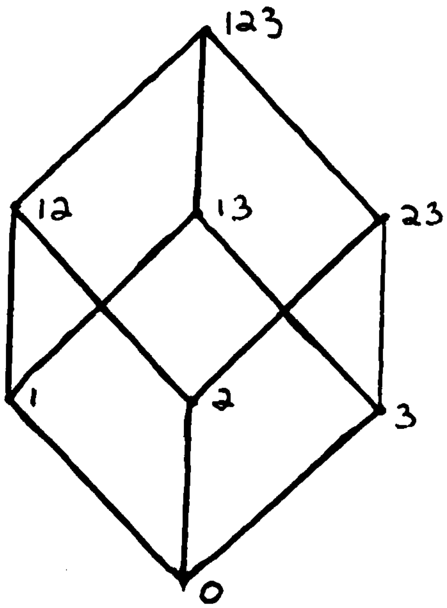


Figure 1.3. Event lattice of example 1.

The readers who are familiar with algebra will recall that this semi-Boolean drawing of the manual is actually an 8-element Boolean structure,

B_8. This is the representation of the power set of a 3-element set. Recall, an n-element set has a 2^n-element power set. Finite Boolean algebras are power sets of a given set. Semi-Boolean algebras are power sets of one or more sets (or maximal elements. Or, if you will, "tests" or "operations.)

Note that in the manual, some elements (or "events") are written above others with lines connecting them. An upward line indicates that the lower element is contained in the upper element. In this manual, there are three events located directly above 0. These are called "atoms" or "outcomes", because they are the most fundamental responses to a test.

It would be wise to digress for just a bit to point out that one of the theorems in Chapter 4 demonstrated that we could write a test as 123, rather than $\{1,2,3\}$, The smallest element as 0, instead of $\{\emptyset\}$, and an ordinary event like $\{1,2\}$ as 12. This notation has been adopted because it is simpler and quicker than using set notation.

When observing this manual drawing (event lattice), one may note that it looks like a cube projected onto a 2-space (the sheet of paper). This is an important observation, since we can learn a lot from considering the manual as a 3-element power set in Euclidean 3-space. For example, the atoms form three orthogonal vectors which span 3-space. The "join" (least upper bound) of two atoms is a plane. And if one joins the third atom to the plane, one gets the entire 3-space, 123. Hence, 12 and 3 are complements since their span (join) is the whole set, and their intersection ("meet" or greatest lower bound) is 0. Likewise for 13 and 2, 23 and 1, and 123 and 0.

We can label the manual using Boolean notation and derive the following manual.

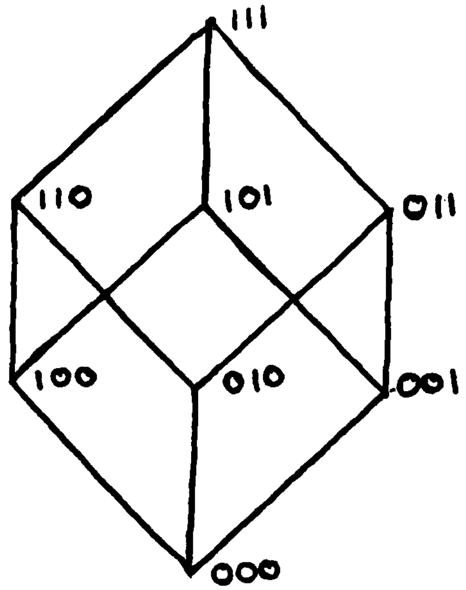


Figure 1.4. Event lattice using Boolean notation.

In this notation, we fix the position of the atoms (in this case, 1 in the first column, 2 in the second column, and 3 in the third), and if the event contains that atom, we write a "1" in the corresponding column. We write a "0" in that column if it does not contain the atom. Here, "contain" means that the atom is less than or equal to the event, or in set notation, the atom is contained in the set.

This notation is useful because most computer languages deal well with binary values. In fact, the computer has been used in such a manner so that conditions which are tedious to check by hand can be rapidly performed by the computer. See the Appendix for an example of such a program.

Recall that in Chapter 2, we proved that the direct product of two manuals is a manual. B_8 is a direct product of B_4 and B_2. B_4 in turn is

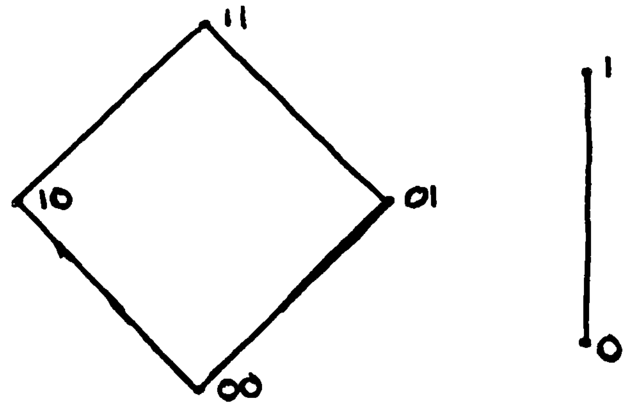


Figure 1.5. B_4 andB_2.

a direct product of B_2 and B_2. So from this, we see that $B_8=B_2xB_2xB_2$.

We have carried our discussion of this classical manual about as far as we can, without discussing weights and pure states. We can define a probability or "weight" distribution assuming the probability that the buoy will not be spotted is 1/2. Assuming that the buoy may be anywhere in the 202.5 degree visible range, the probability of event 1 is 1/2x4 compass points/18 compass points = 2/18; and the probability of event 3 is 1/2 x 14/18 = 7/18. The drawing of the weights on the event lattice is found in Figure 1.6.

In a classical system, a pure state or extreme point of the set of weights is simply a case in which an outcome has probability 1 and the other outcomes have probability 0. For example, if the event 2 has probability 1, then the probability density distribution is as in Figure 1.7. This case corresponds to the physical situation in which it is known that the buoy is not in visual range. If the "question" "is the buoy out of range?" is asked, the answer is "yes" and the physical "entity" consisting of the ship and the buoy is characterized by the "property" that the buoy is not visible from the port side.

The other pure states are the cases where either the buoy is dead ahead (probability of event 1 is 1, probability of both events 2 and 3 are 0),

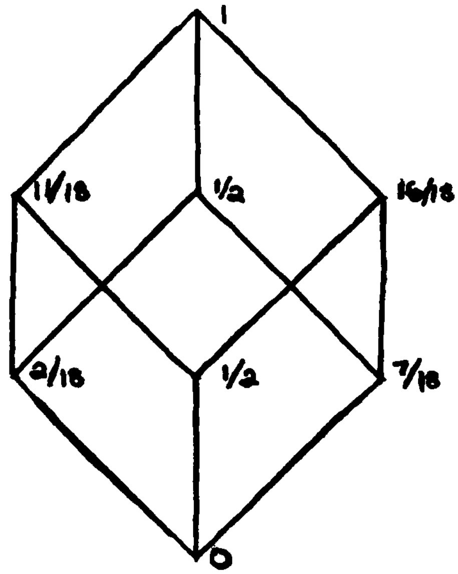


Figure 1.6. A sample state on the event lattice.

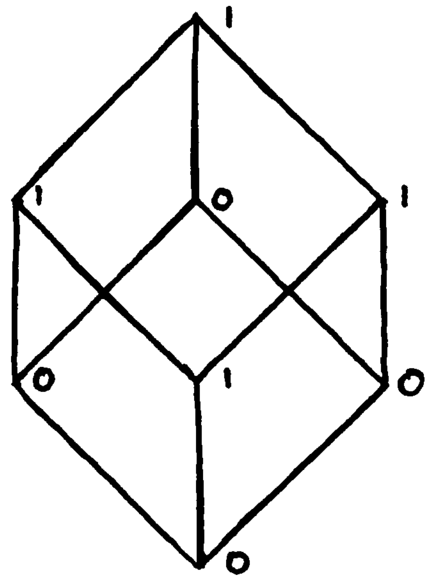


Figure 1.7. A pure state.

or the buoy is on the port quarter. If the latter is the case, then the property of the buoy being on the port quarter (3) is said to be actual. Otherwise, it is said to be potential.

The weights x_1, x_2, x_3, corresponding to the weights of the events 1, 2, and 3, respectively, represent the "state" of the physical situation satisfying the equation

$$x_1 + x_2 + x_3 = 1$$

subject to the constraint that x_i 0 for 1 i 3. The solution set in 3 variables is represented by points on the plane $x_1 + x_2 + x_3 = 1$ in the first octant. The solutions therefore form a triangle which is a convex set whose extreme points are (1,0,0), (0,1,0), and (0,0,1). The sample state given in Figure 1.6 is a convex linear combination of these three pure states. Figure 1.8 is a drawing of the convex solution set.

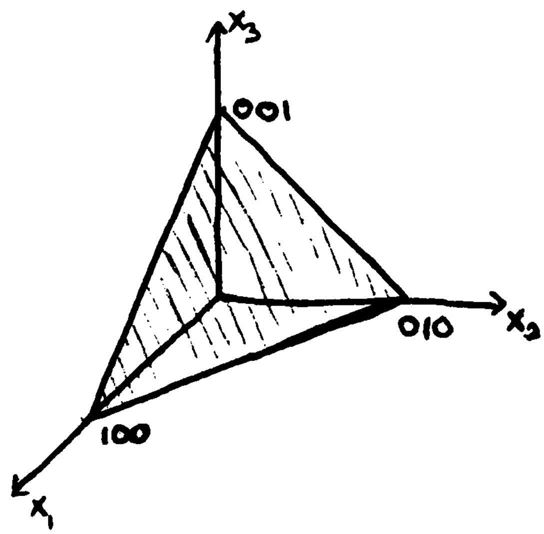


Figure 1.8. Convex solution set.

Example 2. The example which we are about to consider is a semi-classical manual. Recall that a semi-classical manual is one in which the intersection of every pair of tests is 0. One such example would be as follows:

- a means the car is red
- $a^{\perp}$ means the car is not red
- 1_a means that the car is either red or not red
- b means that it is hot outside
- $b^{\perp}$ means that it is not hot outside
- 1_b means that it is either hot or not hot outside

The Greechie diagram looks like the following:

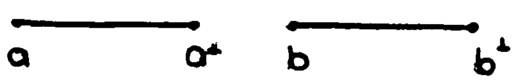


Figure 2.1. Greechie diagram of a semi-classical manual.

Note that the diagram looks just like two Greechie diagrams for a classical manual. This is because in a semi-clasical manual, the tests are unrelated. Thus, in the drawing of the manual, we have a semi-Boolean algebra which appears to be two B_4's attached at 0.

This manual is a dacification of Z_2. Recall that dacification is a ghosting of each event in the dominating set (set of tests), ghosting being the process in which a unique atom is joined with each event contained in a principal ideal. Z_n is the semi-classical manual which has the integers 1 through n as both the atomic set (set of atoms) and the dominating set.

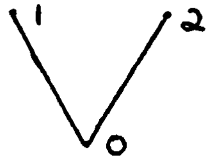


Figure 2.3. Z_2.

The only events which are op in this manual are 1_a and 1_b. If we construct the "op logic" (called "logic", for short) we tie together equivalent events in the sense of being equivalent modulo op. The logic looks like this:

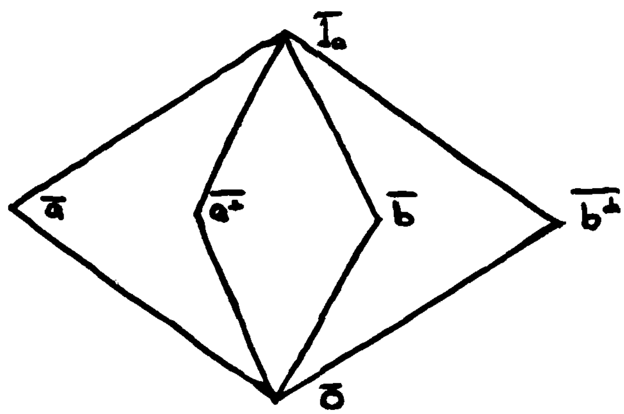


Figure 2.4. A semi-classical logic.

This structure is known as OM_{96}, the "OM" being derived from orthomodular, a property which is similar to, but weaker than, the distributive property. The orthomodular law says that $a \vee (a^\perp \wedge b) = b$ for all a and b such that $a \leq b$. It is not difficult to demonstrate that the distributive property does not hold on the logic.

$\bar{a} \vee (\overline{a^\perp} \wedge \bar{b}) = \bar{a} \vee \bar{0} = \bar{a}$, but

$(\bar{a} \vee \overline{a^\perp}) \wedge (\bar{a} \vee \bar{b}) = \bar{1}_a \vee \bar{1}_a = \bar{1}_a$. Therefore, $\bar{a} \vee (\overline{a^\perp} \wedge \bar{b}) \neq$

$(\bar{a} \vee \overline{a^\perp}) \wedge (\bar{a} \vee \bar{b})$, and the distributive property does not hold.

Hughes produced an identical structure in his October 1981 issue of Scientific American.[4] Instead of heat and car color though, he used spin up and spin down in the x and y directions. These two quantities, x spin and y spin, are not compatible observables. That is to say, the accuracy of the measurement of one effects the accuracy of the measurement of the other. This is the difficulty in quantum physics which all non-compatible observables share.

Example 3. The manual which is about to be examined is the first manual with which we have dealt that is neither classical nor semi-classical. It is non-classical, which means that it has more than one test, and has at least two tests whose intersection is not equal to zero. As was promised in Example 1, this problem is a more complicated version of the same problem. Example 4 will even be more complex.

We will keep the test 123 in this manual, with the events designating exactly the same as in Example 1. In addition, we will add the test 156, corresponding to adding a starboard lookout to the watch section. The meaning of the added events are as follows:

5 if the buoy is located two points off the starboard bow to dead astern (22.5 degrees to 180 degrees relative, hereafter known as the starboard quarter)
6 if the starboard lookout does not see the buoy
15 if the starboard lookout sees the buoy (known as the starboard side)
16 if the buoy is not on the starboard quarter
56 if the buoy is not dead ahead
156 if the starboard lookout looks for the buoy

The schematic diagram for the manual is as follows:

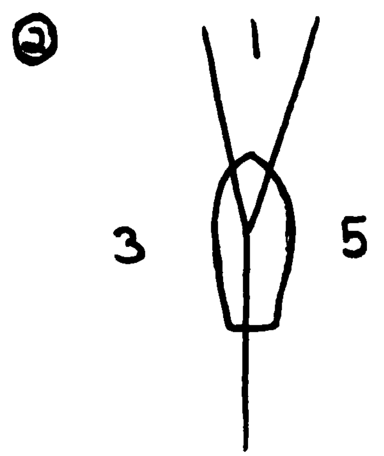
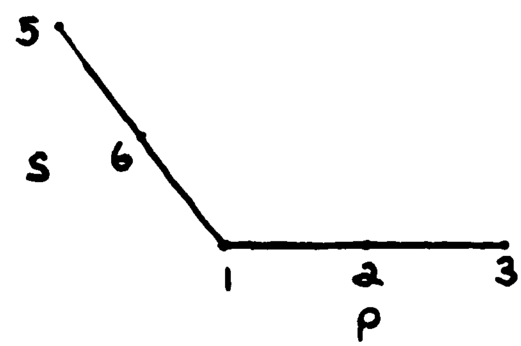


Figure 3.1. Two lookout problem.

In the Greechie diagram one should note that there are two line intersecting at a single point. This indicates that the two tests share a common outcome. Both the Greechie diagram and the event lattice are as follows:

Figure 3.2. Greechie diagram of two lookout example.



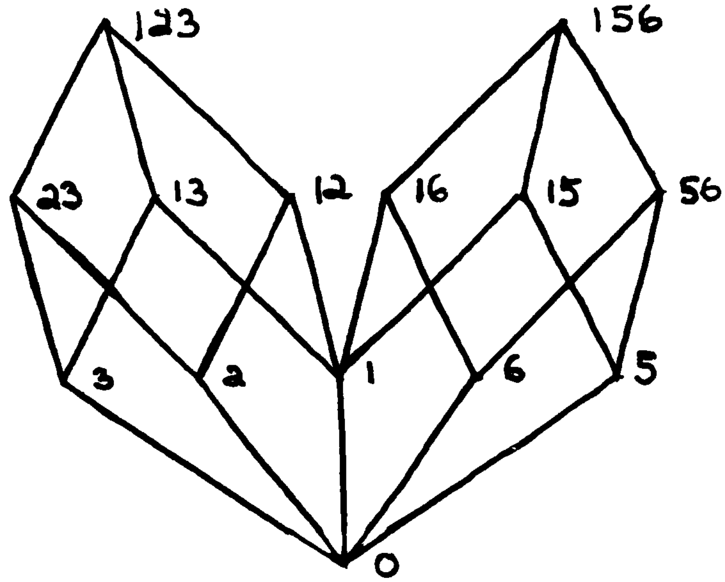


Figure 3.3. Event lattice of this example.

The manual can be drawn in a slightly different manner which makes it easier to identify the op classes.

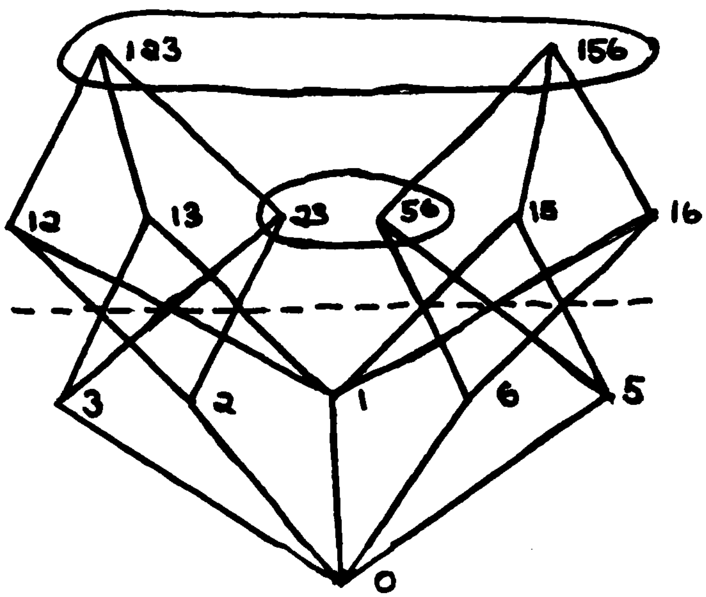


Figure 3.4. Alternate drawing of the manual showing dotted oc reflection line and circled op classes.

As the reader can see in Figure 3.4, the op classes are circled. 123 op 156 and 23 op 56. 123 and 156 both mean that a test was performed, and 23 and 56 both mean that the buoy was not dead ahead. So, in a very real sense, op classes are classes of "equivalent" events.

Note that in this figure, the dotted oc reflection line really acts as a line separating an event from its operational complement. For example, 12 is directly across the line from its operational complement 3, and 123 is as far away from the dotted line as its oc, 0.

When we connect the circled op pairs in Figure 3.4, we end up with the logic in Figure 3.5.

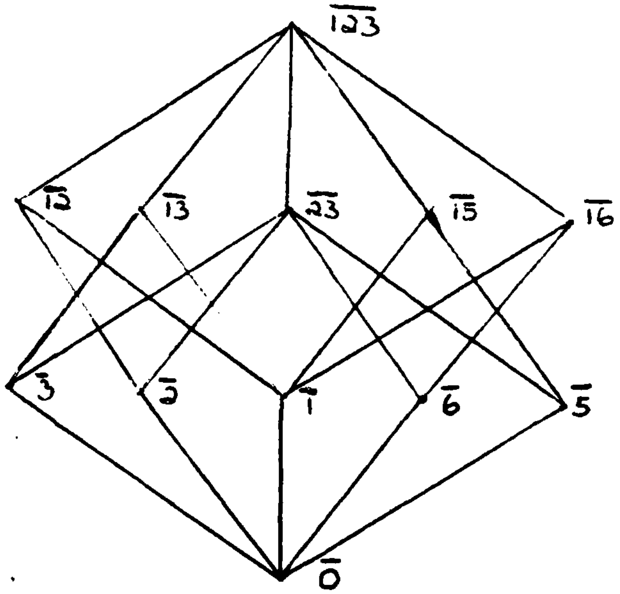


Figure 3.5. The op logic.

However, if we reconfigure it, we come up with the alternate form of the logic found in Figure 3.6.

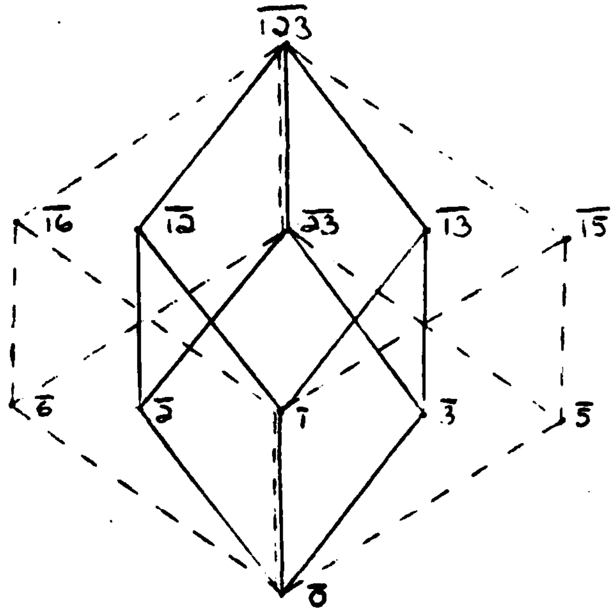


Figure 3.6. The alternate representation of the logic.

This alternate drawing is useful for several reasons. First of all, it gives one a sense of orthogonality, as in Example 1. Secondly, one can see that from this figure, $\overline{123}$ and $\overline{156}$ are simply different coordinate systems for Euclidean 3-space, sharing only the axis $\overline{1}$. Also, $\overline{23}$ and $\overline{56}$ are representations of two sets of vectors spanning the same 2-space. Figure 3.7 shows the same 3-space coordinate systems from a different perspective.

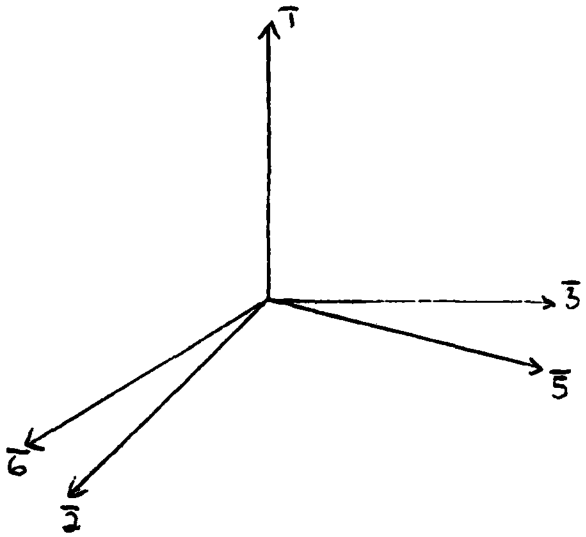


Figure 3.7. Logic in 3-space from a different perspective.

When one notes that Euclidean 3-space is really real-valued 3-dimensional Hilbert space, one wonders what ramifications this may have when placed in the perspective that quantum mechanics was formally derived in terms of Hilbert spaces. This will be discussed more after we have looked at Example 4.

Chapter 6 was spent looking at a little of the work of Aerts and Piron. Their work deals with "questions", "states", and "properties." What we will look at is the property lattice. Recall that a property lattice is ordered by "-<", which is spoken as "stronger than", where a -< b if whenever a is actual, b is actual. We will construct the filters of the primitive properties, since it appears that a manual filter lattice taken form the event lattice is an upside down primitive property lattice. The order on the filter lattice is set containment.

F(1) = {1,12,13,15,16,123,156}
F(2) = {2,12,23,56,123,156}
F(3) = {3,13,23,56,123,156}
F(5) = {5,15,56,23,123,156}
F(6) = {6,16,56,23,123,156}
F(12) = {12,123,156}
F(13) = {13,123,156}
F(23) = F(56) = {23,56,123,156}
F(15) = {15,123,156}
F(16) = {16,123,156}
F(123) = F(156) = {123,156}
F(0) = {1,2,3,5,6,12,13,15,16,23,56,123,156,0}

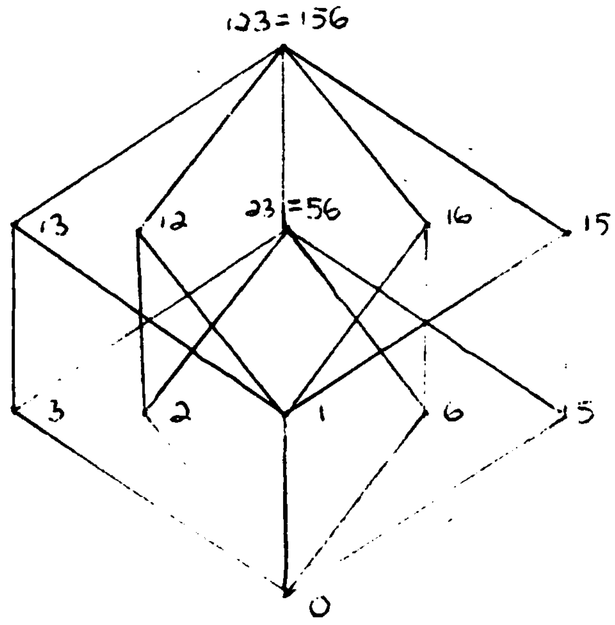


Figure 3.8. Primitive property lattice.

In this case, the primitive property lattice is the same as the logic, though by no means will this always occur. Note that if one turned this structure upside down, one would have the filter lattice. The lattice in Figure 3.8 has been dubbed "the ghost of OM_6", since it is the direct product of B_2 and OM_6.

Suppose, as Aerts suggests, we allow non-primitive properties. These are formed by combining two or more primitive properties using "·" (spoken "and"), which gives us a property which is actual if and only if any of the primitive questions associated with the primitive properties connected by the dot, when tested, would also be true.

Aerts suggests an example in his doctoral thesis to explain his meaning. If we were to test a piece of wood to see if it floats, we would expect that it would. If the test is true , we say it has property a. If we were to test a piece of wood to see if it burns, again we expect it would. If the test is positive, we say it has property b. If we want to designate for that same piece of wood that it will float and it will burn, we designate it a·b. If we tested either property, we would find it to be actual. But what happens if we try to test both properties? If you test a piece of wood to see if it floats, it will get wet and not burn. If you test a piece of wood to see if it will burn, it will probably not float very well. For a·b to be actual, we do not require that a and b both be tested and be found actual. Clearly, you may test one, and as a result of that test, effect the second test.

Recalling Hughes' example fom Scientific American concerning spin up and down in the x and y directions, if a particle is spin up x and spin up y, let us call this property c·d, then whenever one is tested, we should get the expected results. But if we test spin up x, and then test to see if we have spin up in the y direction, we may find that we have spin down in the y direction. This is because testing spin in the x direction effects the spin in the y direction.

Getting back to the original goal of Example 3, let us generate the entire property lattice by generating the remaining manual filters. Note that we would not have a filter like F(1·2), since 1 ∨ 2 exists, and 1 ∧ 2 = 0, and so we would just simply generate F(0). Recall that if the join exists, the meet must also be in the filter.

F(2·5) = {2,5,12,23,56,15,123,156}
F(2·6) = {2,6,12,23,56,16,123,156}
F(3·5) = {3,5,13,23,56,16,123,156}
F(3·6) = {3,6,13,23,56,16,123,156}
F(12·15) = {12,15,123,156}
F(12·16) = {12,16,123,156}
F(13·15) = {13,15,123,156}
F(13·16) = {13,16,123,156}

Our new property lattice looks like the following:

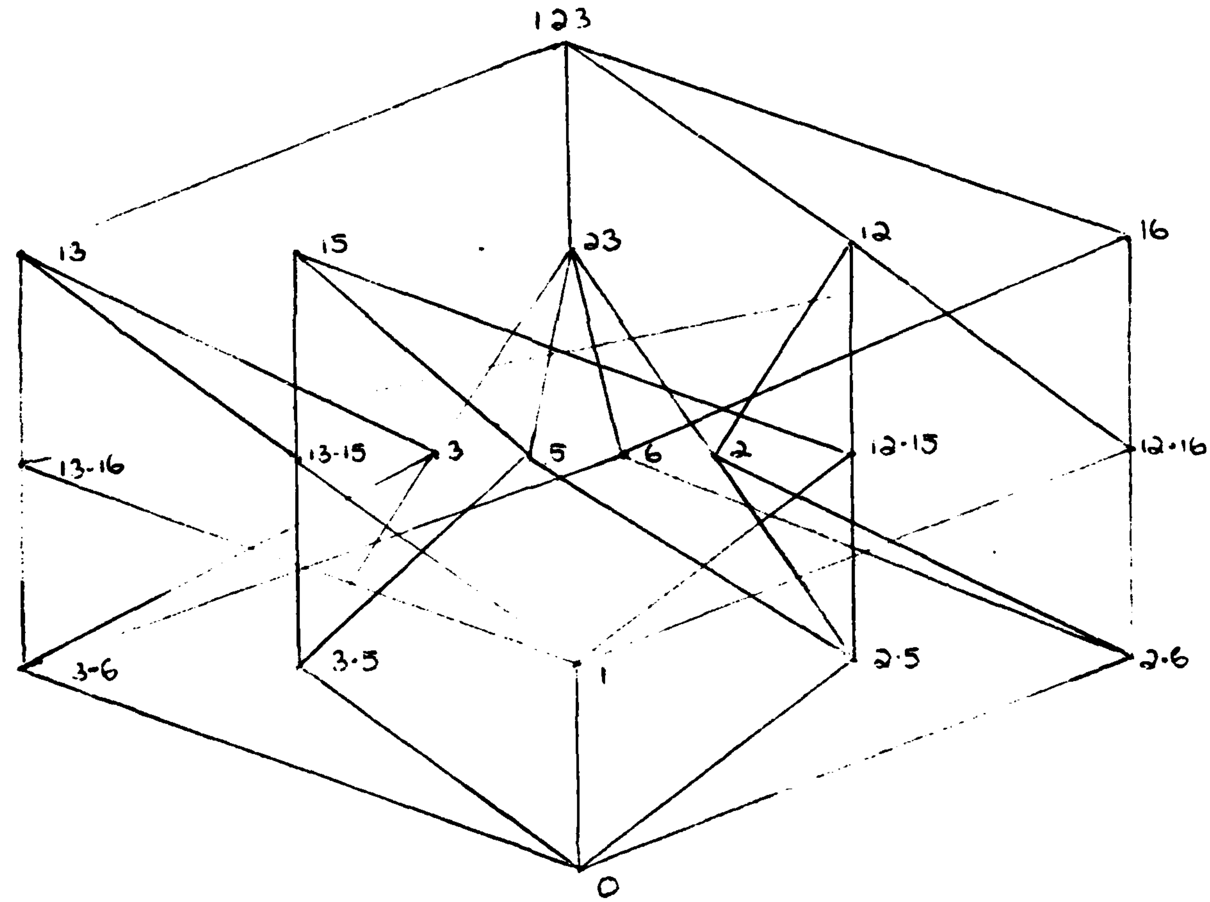


Figure 3.9. Two lookout property lattice.

It is clear that this lattice is a lot more complex than the primitive property lattice. Again, if one inverts the lattice we will be left with the complete manual filter lattice ordered by set containment.

Example 4. As promised, this example is a more complex version of the one found in Example 3. In this case, we post a new lookout, the aft lookout, who can see sector 3 and sector 5, but cannot see sector 1. The following is a list of interpretations of events which are in this manual, but not seen in either example 1 or example 3.

4 if the aft lookout cannot see the buoy
34 if the buoy is not in the starboard quarter
35 if the aft lookout sees the buoy
45 if the buoy is not in the port quarter
345 if the aft lookout is looking for the buoy

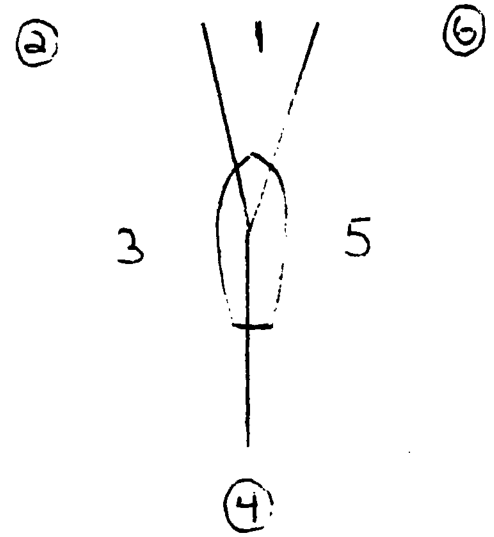


Figure 4.1. Drawing of the lookouts' fields of vision.

When we draw the Greechie diagram ofthe manual, we get the following:

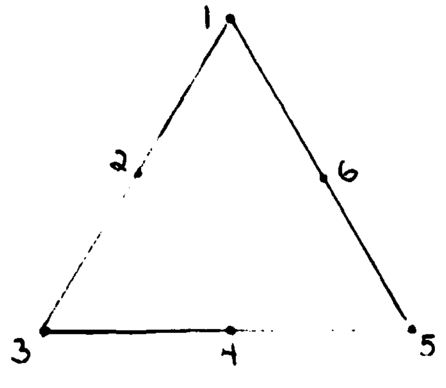


Figure 4.2. Greechie diagram of Wright Triangle.

Because of the Greechie's shape and because of the work of Ron Wright with this manual, it is called the "Wright Triangle." This manual is interesting for many reasons, which we will soon see.

Looking at the drawing of the manual in Figure 4.3, note that in the

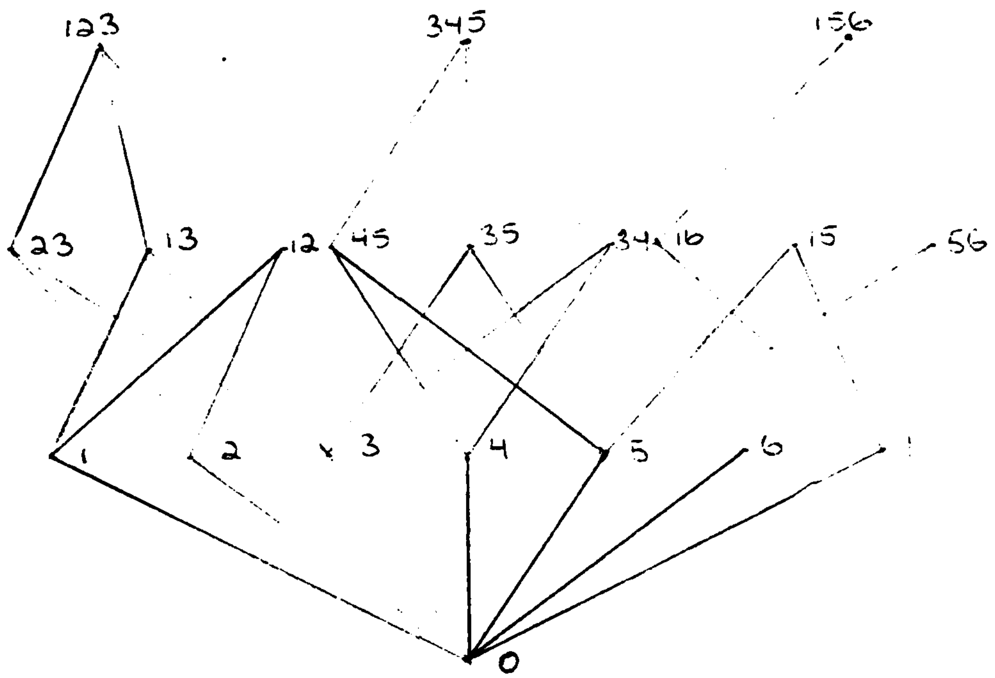


Figure 4.3. Event lattice for the
Wright Triangle.

rows of atoms, 1 is listed twice. This is because the manual wraps around itself. That is to say, it would be more accurate to draw the manual on the surface of either a cylinder or a cone.

One may question at this point how we know that this semi-Boolean algebra is really a manual. There are several ways to verify this. One is to make use of a computer program entitled "MANUAL1." To use this program, we enter into the terminal each of the three tests, one at a time. The computer will take these tests, generate the semi-Boolean algebra, find the relative complements of each event with respect ot the tests in which each is contained, search out op's, and then verify that every possible combination of x oc y, y op z, yields x oc z. In this way, we found that the Wright Triangle is a manual. This program is found in the Appendix.

Another way of verifying that we have a manual is to note that this manual is a dacification of a semi-classical manual. In Chapter 2, it was shown that a dacification is always a manual.

A third way of verifying that we have a manual is by checking that all of our "Z's" are "crossed." This method was devised at the University of Massachusetts at Amherst.[10] It says that one should write an event connected with a straight line to an event oc to it on the other side of the original line. This new event is connected with all events oc to it on the first side of the line. If we form a "Z" which is not crossed, then we do not have a manual.

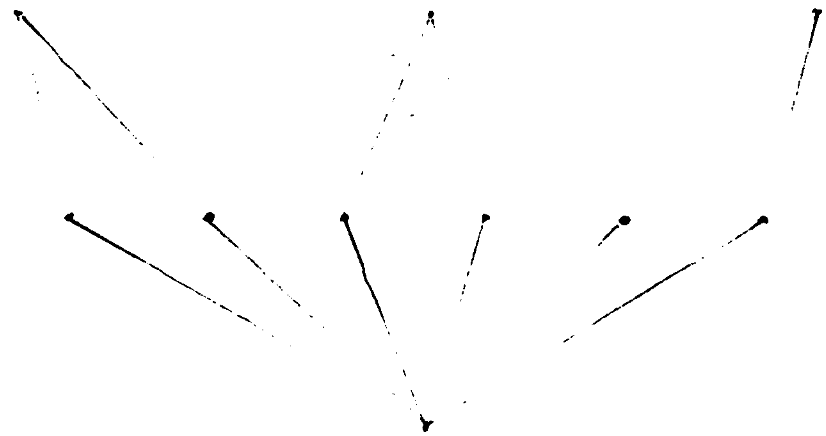


Figure 4.4. Semi-classical manual through which the Wright Triangle was arrived by means of a dacification.

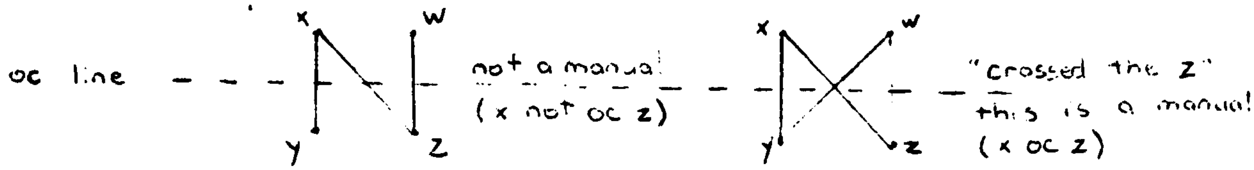


Figure 4.5. Amherst method of checking for a manual.

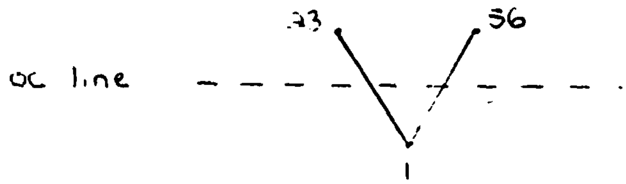


Figure 4.6. Result of applying Amherst method to the Wright Triangle.

Now that we have more than settled that the Wright Triangle is a manual, let us construct a logic. We have the following non-trivial op equivalence classes:

12 op 45, 23 op 56, 34 op 16, and 123 op 345 op 156

If one checks the interpretation of an op pair, for example 12 and 45, one will note that they are identical. Both 12 and 45 mean that the buoy is not on the port quarter. This demonstrates the reason for looking at the logic.

As was mentioned in Chapter 3, the logic may exhibit certain difficulties, and this logic has more than one. Just like in Example 3, the logic is not distributive. But an even more significant problem is that the least upper bound of $\bar{1}$ and $\bar{3}$? One possible candidate is $\overline{13}$, but another possible least upper bound is $\overline{34} = \overline{16}$. This presents some difficulty in working with the logic.

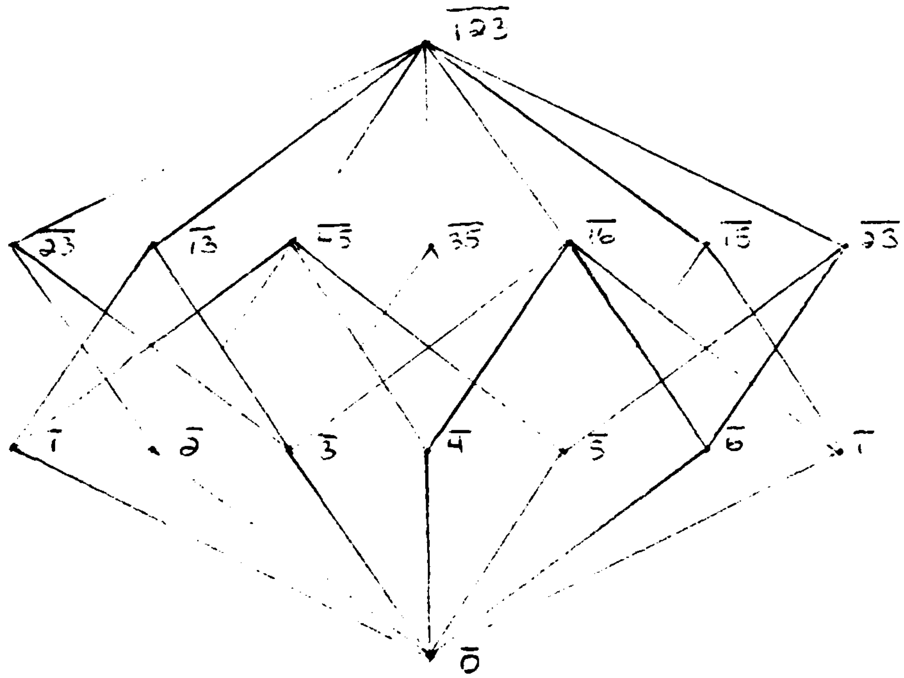


Figure 4.7. The logic.

In Example 3, we were able to represent the logic in Euclidan 3-space. In this example, we cannot. The inability to represent it in Euclidean 3-space, or a finite Hilbert space of 3 dimensions, may imply several things about the present formal development of quantum mechanics. Since quantum mechanics was developed in terms of infinite dimensional Hilbert space, a logic structure such as this which exists in terms of empirical logic may indicate that the Hilbert space structure limits quantum mechanics by not allowing certain systems of tests or certain states. On the other hand, it may also be reasoned that quantum mechanics, by nature, will not allow a logic such as this one.

The next thing that we want to look at with respect to this example is weights. In Chapter 5, we said that a weight is a way of assigning values between 0 and 1 to each event in such a way that the sum of the weights of the atoms in each test is 1, and the weight of an event is equal to the sum of the weights assigned to the atoms which it contains. From these qualifications, then, we can deduce a system of equations designed to yield solutions to the constraints.

If we allow x to mean "the weight assigned to event n", then we have the following system of equations:

$$x_1 + x_2 + x_3 = 1$$

$$x_3 + x_4 + x_5 = 1$$

$$x_1 + x_5 + x_6 = 1$$

and $0 \leq x_i \leq 1$ for $1 \leq i \leq 6$

This system of equations completely describes the set of allowable weights, which when graphed in 6-space forms a convex set. If we could generate the extreme points in some way, then all of the allowed solutions are simply convex linear combinations of the extreme points. We will call the extreme points "pure states." These pure states are of importance also because Aerts has asserted that the atoms of the property lattice correspond to pure states, if the pure states exist.

By borrowing some theorems from the simplex method of linear programming, we have found a technique for solving for the pure states, if they exist. The reader should refer to Chapter 5 for the theorems.

One of the qualifications of this method is that all varialbles be non-negative. We already have this with $0 \leq x_i$ for $1 \leq i \leq 6$. In addition, since each of the weights is non-negative, the first three equations already require $x_i \leq 1$ for $1 \leq i \leq 6$. Thus, we have reduced our system to 3 equations and 6 unknowns, which can be expressed as a 3 x 6 augmented matrix such as the following:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The technique requires us to set 3 columns in the 3 x 6 unaugmented matrix equal to zero. If the remaining 3 x 3 determinant is non-zero, implying linear independence of the remaining three vectors, one then solves the 3 x 3 augmented matrix. If each of the variables in the solution is non-negative, then the solution corresponds to a pure state. If one or more variable is negative, then the solution does not correspond to a pure state. If one sets every possible combination of n - m (number of columns minus number of rows) columns equal to zero, one will find all of the pure states. In this case, as mentioned above, n - m = 3.

Proceeding to solve this problem, we get the following:

<u>Non-zero columns</u>

1,2,3 $\quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad x_1 = x_3 = 1,\ x_2 = -1$

Since x_2 is negative, this does not correspond to a pure state.

1,2,4 $\quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad x_1 = x_4 = 1,\ x_2 = 0$

Since this solution meets all of the conditions, one pure state is $(x_1,x_2,x_3,x_4,x_5,x_6) = (1,0,0,1,0,0)$. Since here $x_2 = x_3 = x_5 = x_6$, we no longer have to look at any combinations with the first and fourth columns as non-zero columns.

1,2,5 $\quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Non-zero columns

$$x_2 = x_5 = 1, \ x_1 = 0$$

Thus we have another pure state — (0,1,0,0,1,0). Note that we no longer have to look at 2-5 combinations.

1,2,6

$$\det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = 0$$

Therefore, this will not yield a pure state.

1,3,4 We skip this since we already solved for a 1-4 combination.

1,3,5

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

$$x_1 = x_3 = x_5 = 1/2$$

This is non-dispersion free, (1/2,0,1/2,0,1/2,0), though it is clearly a pure state. Dispersion free states, states with only 0's and 1's, are associated with classical results. This non-dispersion free pure state is purely a non-classical result.

1,3,6

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad x_3 = x_6 = 1, \ x_1 = 0$$

So our fourth pure state is (0,0,1,0,0,1). There is no need to look at 3-6 combinations again. We can skip 1,4,5 and 1,4,6.

1,5,6

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad x_1 = x_5 = 1, \ x_6 = -1$$

Since we have a negative solution, this does not correspond to a pure state.

2,3,4

$$\det \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

We skip 2,3,5; 2,3,6; and 2,4,5.

2,4,6

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad x_2 = x_4 = x_6 = 1$$

The fifth solution is (0,1,0,1,0,1). We skip 2,5,6.

3,4,5

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad x_3 = x_5 = 1, \ x_4 = -1$$

This does not correspond to a pure state. We skip 3,4,6 and 3,5,6.

Non-zero columns

4,5,6

$$\det \begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

Finally, the process is complete, and we have 5 pure states: (1,0,0,1,0,0), (0,1,0,0,1,0), (0,0,1,0,0,1), (1/2,0,1/2,0,1/2,0), and (0,1,0,1,0,1). From these solutions, all of the weights may be found.

One of the most useful things about this technique is that it is easily adaptable to the computer, since the entire procedure just demonstrated used a simple algorithm. As a result, the process can even be made quicker and easier.

We will now move on to generating the primitive property and property lattices. We will denote 123 by p, 345 by a and 156 by s. This is done for the sake of brevity.

F(0) = {the set of events}
F(1) = {1,12,45,13,15,16,34,4,p,a,s}
F(2) = {2,12,45,23,56,p,a,s}
F(3) = {3,13,23,56,34,16,6,35,p,a,s}
F(4) = {4,34,16,45,12,p,a,s}
F(5) = {5,15,35,45,12,56,23,2,p,a,s}
F(6) = {6,16,34,15,23,p,a,s}
F(23) = F(56) = {23,56,p,a,s}
F(12) = {13,p,a,s}
F(12) = F(45) = {12,45,p,a,s}
F(35) = {35,p,a,s}
F(34) = F(16) = {34,16,p,a,s}
F(15) = {15,p,a,s}
F(p) = F(a) = F(s) = {p,a,s}

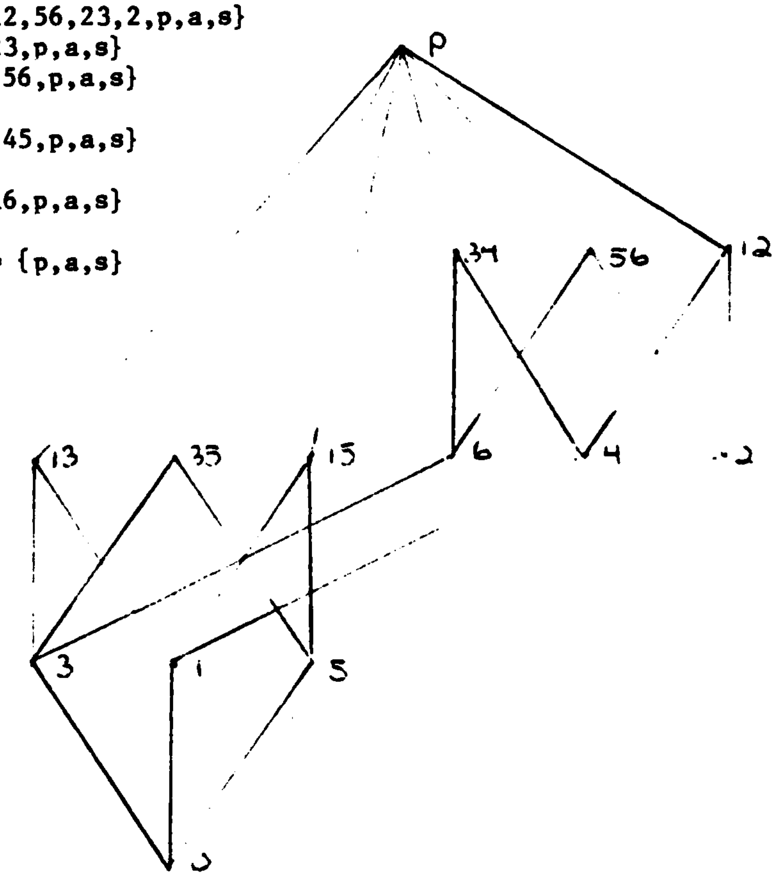


Figure 4.8. The primitive property lattice.

We will now construct the entire property lattice.

F(13·15) = {13,35,p,a,s}
F(15·35) = {15,35,p,a,s}
F(13·15) = {13,15,p,a,s}
F(13·15·35) = {13,15,35,p,a,s}
F(13·34) = {13,34,16,p,a,s}
F(35·23) = {35,23,56,p,a,s}
F(15·12) = {15,12,45,p,a,s}
F(2·4·6) = {2,4,6,12,23,34,45,56,16,p,a,s}

And thus we arrive at the property lattice.

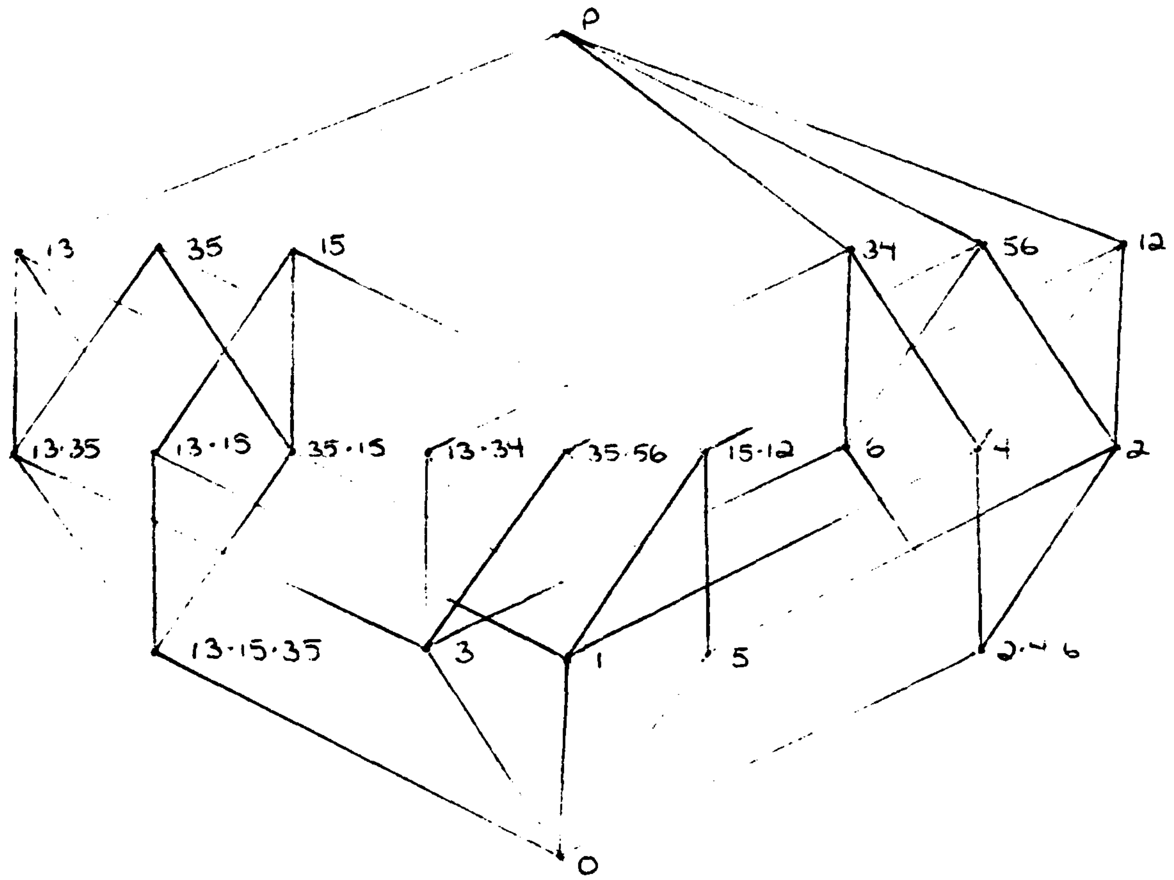


Figure 4.9. Property lattice for the Wright Triangle.

It is proper at this time to note a few things about this property lattice. First of all, 1 -< 4. This makes sense to us logically since whenever the buoy is dead ahead, we would expect the aft lookout to report that he could not see the buoy. But 4 -$\not<$ 1, which also makes sense, since the aft lookout could report not seeing the buoy if the buoy were really not within visual range of any of the lookouts.

In this manual, 2 -< 5, but in the manual in Example 3, 2 -$\not<$ 5. We need to find some explanation for this occurance, since our logical reasoning tells us that when 2 occurs, 5 should also occur. One way of explaining this is to say that the manual in Example 3 simply does not have enough information, or tests, to describe the system. This is somewhat supported by the fact that the manual in this example, having one more test, accurately describes the relationship between 2 and 5.

Another explanation is that the manual in Example 3 does not distinguish between the events 2 and 3, and between the events 5 and 6, when one looks at the systems of equations which describe the states. So, mathematically, there is no reason to say $2 \prec 5$, since clearly $3 \not\prec 5$, and 2 and 3 can, in terms of the system of equations, be interchanged.

Finally, another possible explanation is that in the property lattice, 2 and $2 \cdot 5$ are essentially the same property, since $2 \cdot 5$ means that the port lookoutsees nothing and the buoy is on the starboard quarter. If we allow 2 to be collapsed onto $2 \cdot 5$, to become identical occurances, then we have the desired order in the property lattice of Example 3.

Aerts' has asserted that pure states are identified with the atoms of the property lattice. For this example, we can see that this assertion is true. 1 coorresponds to (1,0,0,1,0,0); 3 corresponds to (0,0,1,0,0,1); 5 corresponds to (0,1,0,0,1,0); $2 \cdot 4 \cdot 6$ corresponds to (0,1,0,1,0,1); and $13 \cdot 15 \cdot 35$ corresponds to (1/2,0,1/2,0,1/2,0).

For our example, we have difficulty in finding an explanation for the last pure state. What it seems to say is that no matter which lookout is asked if he sees the buoy, he will respond that he sees it. It is as if the captain of the ship turns the boat so that the lookout he asks will automatically see the buoy. This is why this state is referred to as being non-classical -- in our ordinary experience, we cannot explain this state.

One way at looking at this state is to say that our system of three tests tells us more about the system ofthe buoy and the lookouts than we need to know. Perhaps we have no right in trying to apply non-classical techniques to this classical problem. It is as if this is a state which knows everything about the system.

One way in which we can deal with this state is to collapse it onto 0. In turn, $13 \cdot 35$ collapses onto 1, $13 \cdot 35$ collapses onto 3, $15 \cdot 35$ collapses onto 5, 13 collapses onto $13 \cdot 34$, 35 collapses onto $35 \cdot 56$, and 15 collapses onto $15 \cdot 15$. This structure, then, stongly resembles the primitive property lattice, except for the fact that this one has the property $2 \cdot 4 \cdot 6$.

It is clear that there is much work left to be done in the area of applying non-classical techniques to classical problems. Not only developing more examples, but also developing a better understanding ofour results. Thus far, people like Foulis, Randall, Piron, and Aerts have accomplished much in the development ofthe theory of empirical science, but have left open the application of this theory to others. It is hoped that these examples suggest some applications, and yet also leave questions in the reader's minds about other applications and other interpretations.

Example 5. Thus far, we have considered four examples of manuals. Perhaps it would be appropriate to introduce a semi-Boolean algebra which is not a manual. Since we have already proven that a semi-Boolean algebra with one or two tests is a manual, we will look at a case with three tests. The Greechie diagram is as follows:

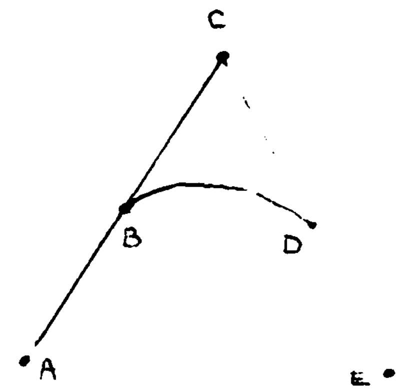


Figure 5.1. Greechie diagram of a semi-Boolean algebra which is not a manual.

In this diagram, the reader will note that this is not the typical type of Greechie diagram, in that one of the tests, ABDE, is represented by a curved line. In many diagrams of this type, there is no way of representing the manual by straight lines. In addition, when there are overlaps of more than one atom, the diagrams are a bit unusual, and we sometimes represent the overlaps by parallel lines drawn closely together.

When we represent this example in an event lattice diagram, we get the drawing in Figure 4.2.

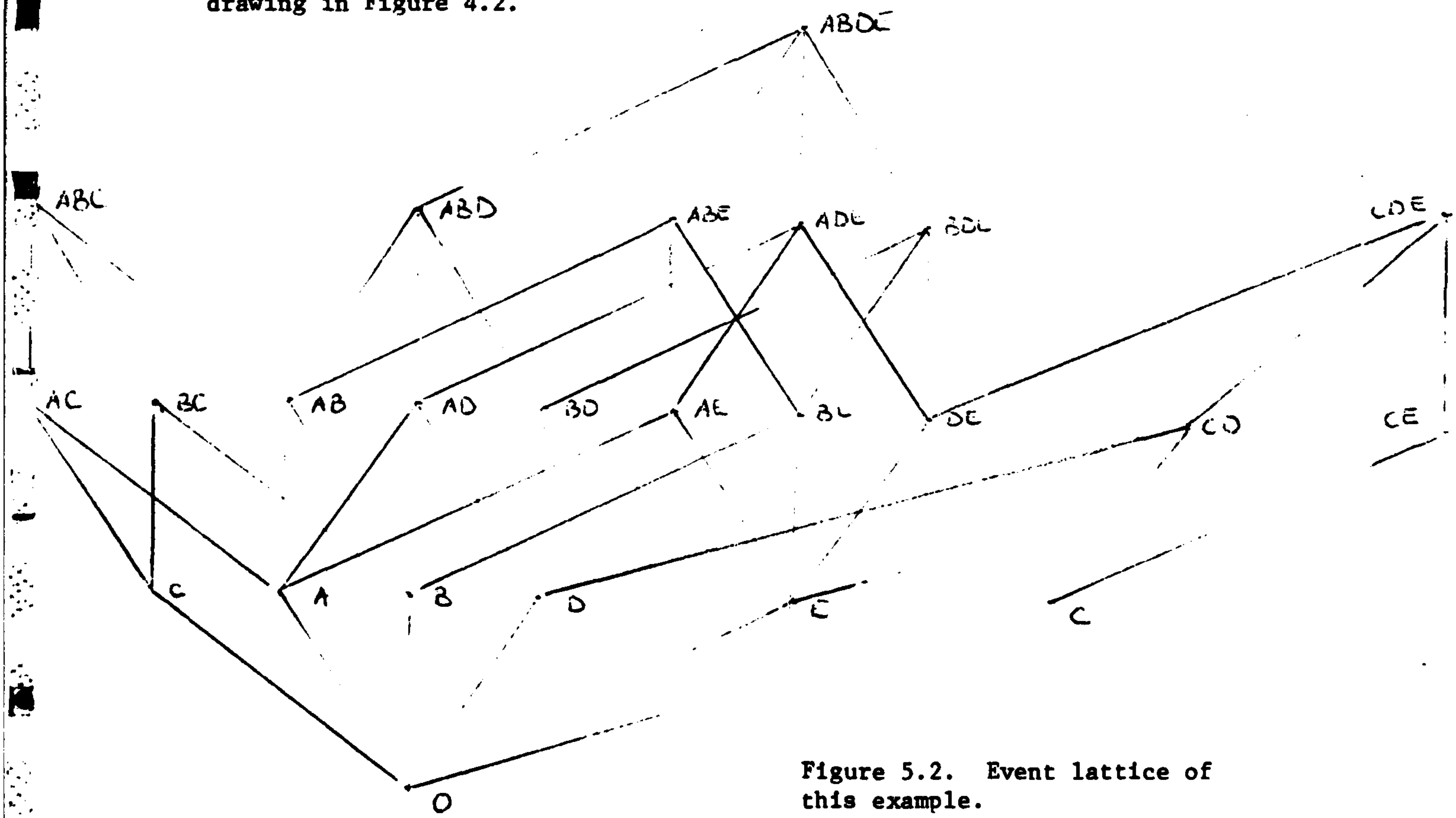


Figure 5.2. Event lattice of this example.

Though this diagram is not ordered as well as the ones in the previous examples, this is not the real key to this not being a manual. Clearly, there are more complicated-looking semi-Boolean diagrams which are manuals. But when we search all possible combinations of op's and oc's, we find that AB oc DE, DE op AB, but clearly AB is not oc to AB. The same is true of C and AB, and C and DE. Therefore, because of these counter-examples, we do not have a manual.

In terms of the method of Z's demonstrated in Example 4, we get the following drawing.

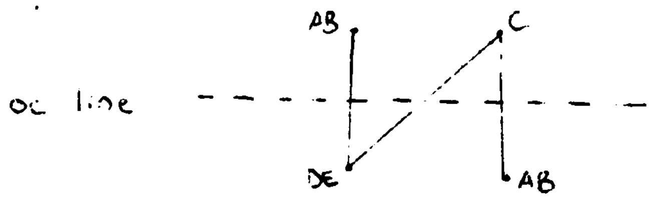


Figure 5.3. Amherst method of checking a manual by Z's.

Since AB is not oc to AB, we cannot connect them across the oc line. This may lead one to ask, "Why is condition M so important?" Since we are talking about events from an event lattice, without condition M, it is possible to have the following: AB occurs (is assigned a weight of 1), which tells us that DE non-occurs (weight of 0). This in turn shows us that C occurs, which implies that AB non-occurs. This contradicts our original assumption of AB occurring. In empirical science, if we perform one test and get a result, we expect that if another test allows that result, then if we had performed that second test originally, we should have gotten that same result.

In some cases, we can add tests to a semi-Boolean algebra to make it into a manual. In this example, we cannot add tests to make it into a manual, because we will never be able to add a test to make AB oc AB. But in the next example, we will be able to make it into a manual by adding a test.

<u>Example 6</u>. This example is called the "hook." It gets its name from the appearance of the Greechie diagram.

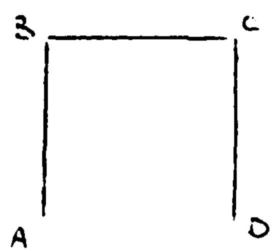


Figure 6.1. Greechie diagram of the "hook."

Our counter-examples to condition M for this example are A oc B, B op D, but A is not oc D. Also, C oc D, C op A, But again, A is not oc D. In order to make A oc D, we need to add the test AD.

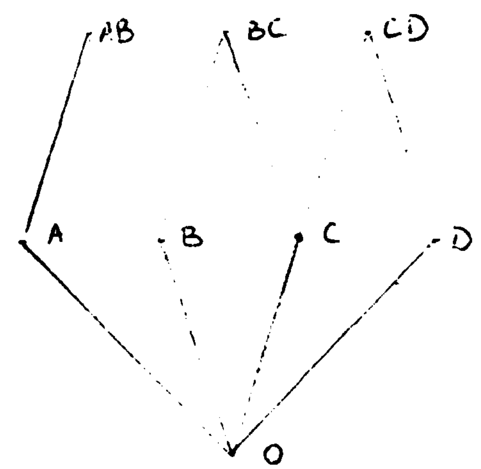


Figure 6.2. Event lattice of the "hook."

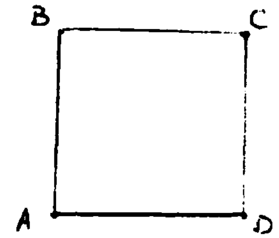


Figure 6.3. Greechie diagram of the "square."

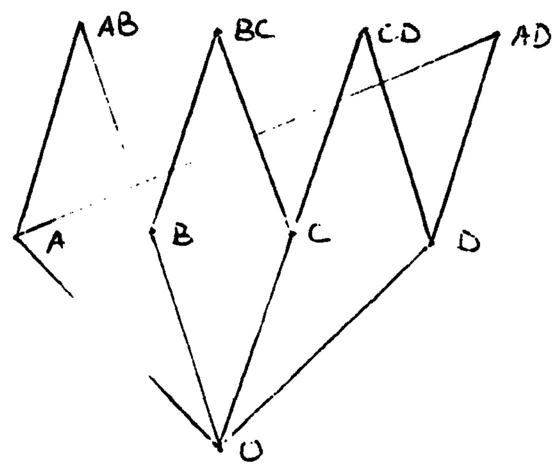


Figure 6.4. Event lattice of the "square."

When we test to see if the new algebra, the "square", is a manual, we find out that it is. Therefore, we have demonstrated a case where the addition of one test made the new algebra into a manual.

Example 7. The purpose of this example is to demonstrate that a manual can have no states, and yet have many filters or properties. The name of this manual is the "windowpane" due to the appearance of its Greechie diagram.

Since a Greechie diagram represents its tests by straight lines, the tests are ABCD, EFGH, IJKL, AEI, BFJ, CGK, and DHL. It is not obvious whether the semi-Boolean algebra satisfies condition M: this is clearly not a dacification, we have more than two tests, and it is not the result of a direct product of manuals. In this case, it is no easy task for one to check all possible combinations for condition M. This is one instance where a computer program saves much time. The windowpane has been verified as a manual, by means of the program MANUAL1, found in the Appendix.

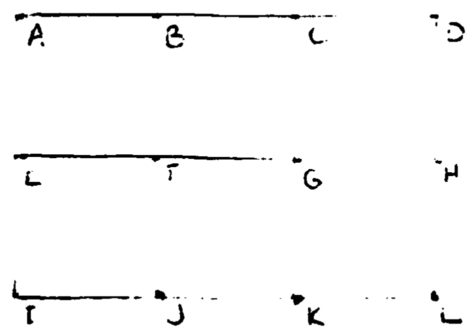


Figure 7.1. Greechie diagram of the windowpane.

Since all of the tests must be assigned a weight of 1, we verify if any states exist. We have 4 tests in vertical columns, and 3 tests in horizontal rows. Thus, if we sum the weights of the tests in a vertical direction we get 4, and if we sum them in a horizontal direction we get 3. As a result, we have found that there is no way to assign weights.

However, filters do exist for this manual. For example, if we generate the filter of A, we get A and all the events which contain it. Also, if A occurs, then an event such as L could occur; and if both A and L occur, then the event FG could also occur. So despite not having any weights, there are many ways of assigning values to events such that every test contains an atom with non-zero probability. We have given an example of a manual filter in which the addition of any event would make the manual filter generate the entire set, called a manual ultrafilter.

Example 8. This example is similar to Examples 1, 3, and 4, in that it has been applied to a navigational system. Refer to Figure 8.1 to see the sectors which divide up the lookout's reports.

In this example, we have two lookouts -- one a port lookout and one a starboard lookout. The port lookout can report 1, 2, 3, 4, 5, or 7 (if the port lookout does not see the buoy). The starboard lookout can report 1, 2, 3, 4, 6, or 8 (if the starboard lookout does not see the buoy). The resulting Greechie diagram is expressed in Figure 8.2. Recall that the side by side lines actually represent an over lap in the Greechie diagram.

When we draw the event lattice, we get a structure as in Figure 8.3.

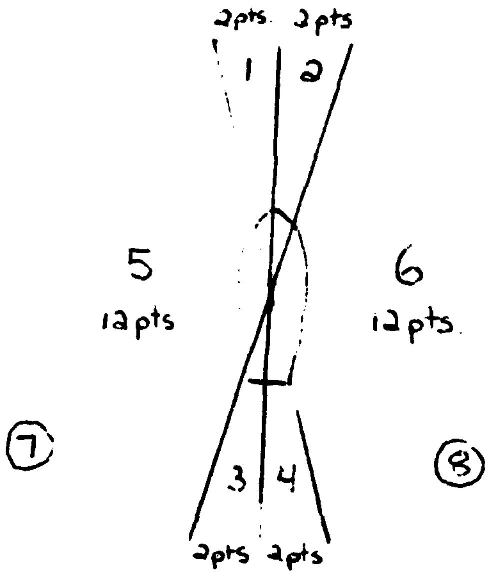


Figure 8.1. Sectors of the lookouts' reports.

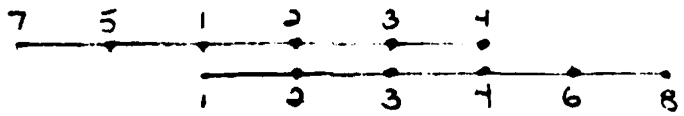


Figure 8.2. Greechie diagram.

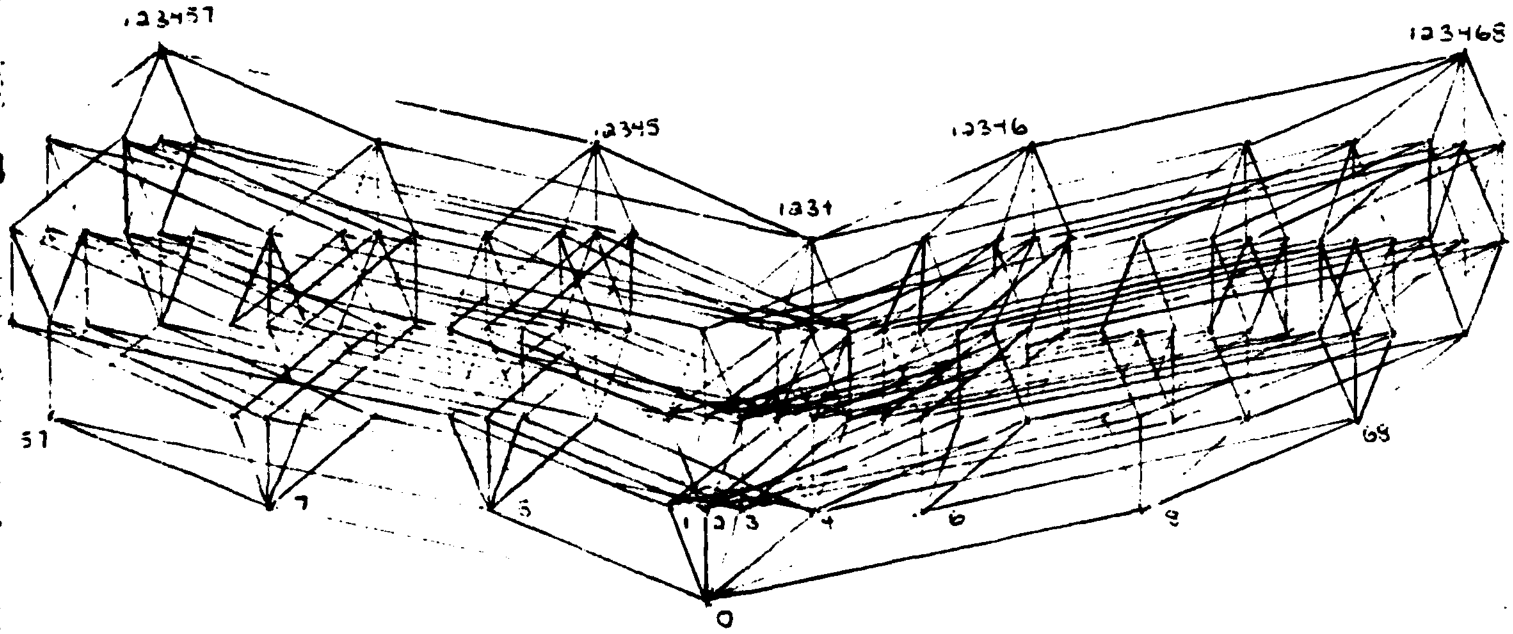


Figure 8.3. Event lattice of 6-sector problem.

One may note that this figure appears to be two 6-dimensional hypercubes overlapping in a 4-dimensional hypercube. This is a semi-Boolean algebra with 112 events. One may note that all of the op pairs can be found within the filters of p∖s and s∖p, which are drawn as B_{16}'s, and are found on the extreme left and the extreme right of the manual. Drawing the logic by overlapping these two filters, we get what is known as OM_{96}. That is, it represents the direct product of OM_6 and B_{16}. But more importantly, OM_{96} is the free algebra generated by two subspaces of a Hilbert space. If one picks out the right events in the logic, one can generate all 96 events (subspaces in a Hilbert manual) by intersection, complementation, subtraction, and spans. It seems then, if we can pick the proper two events, we could generate every other event by asking combinations of these two questions. Perhaps this could be useful in the analysis of certain types of problems which have a similar Greechie diagram, and are constrained by the number of allowed variables.

Conclusions

As was mentioned in the introduction, empirical science was largely brought about as a method of rigorously formalizing quantum physics. In that section, we cited several instances in which the outcomes to quantum mechanical experiments vary significantly from what one would expect to get classically. Motivation was given concerning the historical development of not only this field, but the field of algebras as well.

In the first six chapters, some areas which are of interest to the empirical logician were developed with a certain amount of rigor. We began with the axioms of subtraction algebra, from which we derived many theorems pertinent to later theorems. We defined meets, joins, and relative complements in terms of this algebra. In addition, we established a constant o and a partial order.

When we moved onto Chapter 2, we proved the equivalence of subtraction algebras and semi-Boolean algebras. We then defined operational complements and operational perspectives, from which we set down the condition which makes a semi-Boolean algebra into a manual. It is this structure in which empirical scientists are interested. From there we defined special kinds of manuals, and established conditions under which certain semi-Boolean algebras were assured to be manuals.

In Chapter 3, we demonstrated that op was an equivalence relation, and set up an order diagram on equivalence classes of events. A partial order was established, and various theorems of properties on the logic were set down and proven.

Chapter 4 accomplished two things. First of all, through a version of Stone's Representation Theorem, we showed that a semi-Boolean algebra was the equivalent to an algebra of sets of atoms contained in each event. Secondly, we explained a computer program written to generate the entire semi-Boolean algebra from the tests, and then checked it to see if the manual conditions were satisfied.

Weights formed the primary concern in Chapter 5. The chapter began by defining weights, and then defined states and pure states. Since one can find all states from all of the pure states, it was useful to have demonstrated here a technique - easily adaptable to the computer - which would find all of the pure states.

Chapter 6 was taken from Aert's doctoral thesis The One and the Many. He described and developed many empirical techniques primarily to uses in his study of quantum physics. After examining his lattices, we found that they were related to the op logics of Foulis and Randall. We later applied these property lattices to some of the examples.

The examples were all found in Chapter 7, the last chapter. Here, we demonstrated a cross section of semi-Boolean algebras which were representative of the various properties discussed in the previous chapters. Of special note were the four examples related to the problem of navigation in which lookouts were used who were perhaps unable to see all the way around the ship. With many variations on this set-up, we were able to demonstrate how empirical techniques would be applied to this problem.

The point that needs to be driven home here is that this is simply one area, outside of quantum physics, in which techniques originally developed to use in quantum logic could be applied. Many other areas seem wide open for use of these techniques. Take for example opinion polls, which have gained popularity in recent years. An opinion poll consists of questions or tests, usually with finite outcome sets in each test. Perhaps applying empirical techniques defined in this paper would reveal things about the opinion poll which will lead to a more accurate interpretation of the results.

The potential for applications, at this point, is unlimited. One simply needs to be able to envision one's area of interest as a system of tests with associated outcome sets. As developments in empirical science continue, interpretation of areas in which empirical techniques have been applied will increase, and perhaps lead to significant understanding of things previously hidden.

Empirical science is potentially a great avenue of understanding and interpretation for all sciences, and many areas outside of science.

ENDNOTES

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[2]Ibid., "Ortho-implication Algebras," Studia Logica, XXXV, 2 (1976).

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[8]C. Piron, Seminar on Theoretical Physics (Amherst, Mass., 1981).

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Appendix

This Appendix contains the computer program, MANUAL 1, which is used to verify that the manual condition is satisfied. Pages A-2 through A-6 has a listing of this Fortran program. Page A-7 has a sample run of the program for the "Wright Triangle." (See Example 4 in Chapter 7.) Pages A-8 through A-10 is the result of a sample run of the "windowpane." (See Example 7 in Chapter 7.) And finally, pages A-11 and A-12 represent a sample run of Example 5 in Chapter 7, a semi-Boolean algebra which is not a manual.

```
**** INPUT DATA ****

PRINT,'INPUT NUMBER OF TESTS'
INPUT, NTESTS
PRINT,'INPUT TESTS ALPHANUMERICALLY, ONE AT A TIME'

**** INITIALIZATION ****

NEV=0
KTESPOS=0
NTESTS=0
CHARACTER*36 CTEST(20)
DIMENSION MTEST(20)
DIMENSION NEVENT(1000)
DIMENSION NEVOC(1000)
DIMENSION NPERP(1000)
DIMENSION NPOS(25)
CHARACTER CA*36
CHARACTER CB*36
CHARACTER CC*36
CHARACTER CKEY*36
DIMENSION NCOUNTEX1(11)
DIMENSION NCOUNTEX2(11)
DIMENSION NCOUNTEX3(11)
DIMENSION NOP1(100)
DIMENSION NOP2(100)
DIMENSION NCOHEX1(11)
DIMENSION NCOHEX2(11)
CKEY='ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789'
NRSHIF=SHIFTL(1,35)

**** POWER SET GENERATION *****

DO 1 I=1,NTESTS
INPUT, CTEST(I)
J=1
K=1
2 IF (CKEY(K:K) .EQ. (CTEST(I)(J:J))) THEN
NPOS(J)=SHIFTR(NRSHIF,K-1)
K=K+1
J=J+1
ELSE
K=K+1
IF (CTEST(I)(J:J) .EQ. ' ') THEN
GO TO 91
ENDIF
END IF
GO TO 2
91 [illegible]J-1
MSHIFLN=SHIFTL(1,M)
NTESTS=NTESTS+1
[illegible]
[illegible]
DO 3 [illegible]MSHIFLN-1
[illegible]
NEV=NEV+1
NEVENT[illegible]
[illegible] IF [illegible] THEN
```

```
NEVENT(NEV)=NEVENT(NEV)+NPOS(K)
K=K+1
ELSE
K=K+1
END IF
IF (K .LE. M) GO TO 4
3 CONTINUE
1 CONTINUE

****   GENERATE EVENT OC WRT GIVEN TEST     ****
KT=1
DO 5  I=1,NEV
IF (I .LE. MTEST(KT)) THEN
NEVOC(I)=EOR(NEVENT(I),NEVENT(MTEST(KT)))
ELSE
KT=KT+1
NEVOC(I)=EOR(NEVENT(I),NEVENT(MTEST(KT)))
ENDIF
5 CONTINUE

****   GENERATE EVENT PERP     ****

DO 6 I=1,NEV
NPERP(I)=NEVOC(I)
DO 7 J=1,NEV
IF (NEVENT(I) .EQ. NEVENT(J)) THEN
NPERP(I)=OR(NPERP(I),NEVOC(J))
ENDIF
7 CONTINUE
6 CONTINUE

****   FIND AND RECORD OP PAIRS    ****

NTOTOPS=1
DO 8 I=1,NEV
DO 9 J=I,NEV
IF (NEVOC(I) .EQ. NEVOC(J)) THEN
IF (NEVENT(I) .NE. NEVENT(J)) THEN
DO 10 K=1,NTOTOPS
IF (NTOTOPS .EQ. 1) GO TO 11
IF (NEVENT(I) .EQ. NOP1(K)) THEN
IF (NEVENT(J) .EQ. NOP2(K))   GO TO 9
ENDIF
IF (NEVENT(I) .EQ. NOP2(K)) THEN
IF (NEVENT(J) .EQ. NOP1(K)) GO TO 9
ENDIF
10 CONTINUE
11 NOP1(NTOTOPS)=NEVENT(I)
NOP2(NTOTOPS)=NEVENT(J)
NTOTOPS=NTOTOPS+1
ENDIF
ENDIF
9 CONTINUE
8 CONTINUE
DO 41 I=1,NTOTOPS
NOP1(I+NTOTOPS)=NOP2(I)
NOP2(I+NTOTOPS)=NOP1(I)
41 CONTINUE
```

```
**** CHECK MANUAL CONDITION ****

I1=0
DO 12 I=1,NEV
DO 13 J=1,2*NTOTOPS
IF (NEVOC(I) .EQ. NOP1(J)) THEN
DO 14 K=1,NEV
IF (NEVENT(K) .EQ. NEVENT(I)) THEN
IF (NEVOC(K) .EQ. NOP2(J)) GO TO 13
ENDIF
14 CONTINUE
I1=I1+1
NCOUNTEX1(I1)=NEVENT(I)
NCOUNTEX2(I1)=NEVOC(I)
NCOUNTEX3(I1)=NOP2(J)
IF (I1 .EQ. 11) GO TO 15
ENDIF
13 CONTINUE
12 CONTINUE

**** MANUAL PRINT STATEMENT ****

15 IF (I1 .EQ. 0) THEN
PRINT, 'THIS IS A MANUAL'
ELSE
PRINT, 'THIS IS NOT A MANUAL'
ENDIF

**** CHECK COHERENT CONDITION ****

I2=0
DO 16 I=1,NEV
DO 17 J=1,NEV
IF (OR(NEVENT(J),NPERR(I)) .EQ. NPERR(I)) THEN
IF (OR(NEVOC(J),NEVENT(I)) .NE. NEVOC(J)) THEN
DO 18 K=1,NEV
IF (NEVENT(J) .EQ. NEVENT(K)) THEN
IF (OR(NEVOC(K),NEVENT(I)) .EQ. NEVOC(K)) GO TO 17
ENDIF
18 CONTINUE
DO 75 K=1,I2
IF (NCOHEX1(K) .EQ. NEVENT(J)) THEN
IF (NCOHEX2(K) .EQ. NEVENT(I)) GO TO 17
ENDIF
75 CONTINUE
I2=I2+1
NCOHEX1(I2)=NEVENT(J)
NCOHEX2(I2)=NEVENT(I)
IF (I2 .EQ. 11) GO TO 19
ENDIF
ENDIF
17 CONTINUE
16 CONTINUE

**** COHERENT PRINT STATEMENT ****

19 IF (I2 .EQ. 0) THEN
```

```
PRINT, 'THIS IS COHERENT'
ELSE
PRINT, 'THIS IS NOT COHERENT'
ENDIF
PRINT,

****   EVENT, OC, AND PERP PRINT OPTION      ****

PRINT,'TYPE IN 1 IF YES'
PRINT,'DO YOU WANT TO PRINT OUT EVENTS, OCS, AND PERPS'
INPUT,K1
IF (K1 .NE. 1) GO TO 60
PRINT,
PRINT 54,'EVENT','OC','PERP'
PRINT 54,'*****','**','****'
54 FORMAT(1X,A5,15X,A2,18X,A4)
DO 20 I=1,NEV
CALL TRANS(NRSHIF,CKEY,NEVENT(I),CA)
CALL TRANS(NRSHIF,CKEY,NEVOC(I),CB)
CALL TRANS(NRSHIF,CKEY,NPERP(I),CC)
PRINT 50, CA,CB,CC
50 FORMAT(1X,A19,2X,A19,2X,A36)
20 CONTINUE
PRINT,

****   OP PRINT OPTION      ****

60 PRINT,'DO YOU WANT TO PRINT OUT OP PAIRS'
INPUT,K2
IF (K2 .NE. 1) GO TO 61
PRINT,
PRINT 55,'OP1','OP2'
PRINT 55,'***','***'
55 FORMAT(1X,A3,17X,A3)
DO 21 I=1,NTOTOPS
CALL TRANS(NRSHIF,CKEY,NOP1(I),CA)
CALL TRANS(NRSHIF,CKEY,NOP2(I),CB)
PRINT 51,CA,CB
51 FORMAT(1X,A19,2X,A19)
21 CONTINUE
61 IF (I1 .EQ. 0) GO TO 62

****   COUNTER-EXAMPLE TO CONDITION M PRINT OPTIO      ****

PRINT,'DO YOU WANT TO LIST THE COUNTEREXAMPLES TO CONDITION M'
INPUT,K3
IF (K3 .NE. 1) GO TO 62
PRINT
PRINT,'X OC Y, Y OC Z, BUT ...'
PRINT 53,'X','Y','Z'
PRINT 53,'*','*','*'
53 FORMAT(1X,A1,19X,A1,19X,A1)
DO 22 I=1,I1
CALL TRANS(NRSHIF,CKEY,NCOUNTX(I),CA)
CALL TRANS(NRSHIF,CKEY,NCOUNTY(I),CB)
CALL TRANS(NRSHIF,CKEY,NCOUNTZ(I),CC)
22 PRINT 50,CA,CB,CC
62 IF (I2 .EQ. 0) GO TO 63
```

```
PRINT,
**** COUNTER-EXAMPLE TO COHERENCE PRINT OPTION ****
PRINT,'DO YOU WANT TO LIST COUNTEREXAMPLES TO COHERENCE'
INPUT, K4
IF (K4 .NE. 1) GO TO 63
PRINT,
PRINT,'X CONTAINED IN Y PERP, BUT X NOT PERP TO Y'
PRINT 56, 'X','Y'
PRINT 56, '*','*'
56 FORMAT(1X,A1,19X,A1)
DO 23 I=1,I2
CALL TRANS(NRSHIF,CKEY,NCOHEX1(I),CA)
CALL TRANS(NRSHIF,CKEY,NCOHEX2(I),CB)
23 PRINT 51,CA,CB
63 END
**** CONVERSION FROM BINARY TO ALPHANUMERIC SUBROUTINE ****
SUBROUTINE TRANS(NRSHIF,CKEY,NCHAR,CCHAR)
CHARACTER CCHAR*36
CHARACTER CKEY*36
DO 91 I=1,36
91 CCHAR(I:I)=' '
J=1
DO 90 I=1,36
IF (AND(NCHAR,SHIFTR(NRSHIF,I-1)) .NE. 0) THEN
CCHAR(J:J)=CKEY(I:I)
J=J+1
ENDIF
90 CONTINUE
END
```

```
INPUT NUMBER OF TESTS? 3
INPUT TESTS ALPHANUMERICALLY, ONE AT A TIME? 123
? 345
? 156
THIS IS A MANUAL
THIS IS NOT COHERENT

TYPE IN 1 IF YES
DO YOU WANT TO PRINT OUT EVENTS, CCS, AND PERPS? 1

EVENT                    CC                       PERP
*****                    **                       ****
                         123                      123456
3                        12                       1245
2                        13                       13
23                       1                        1
1                        23                       2356
13                       2                        2
12                       3                        3
123
                         345                      123456
5                        34                       1346
4                        35                       35
45                       3                        3
3                        45                       1245
35                       4                        4
34                       5                        5
345
                         156                      123456
6                        15                       15
5                        16                       1346
56                       1                        1
1                        56                       2356
16                       5                        5
15                       6                        6
156

DO YOU WANT TO PRINT OUT OP PAIRS? 1

OP1                      OP2
***                      ***
23                       56
12                       45
123                      345
123                      156
34                       13
345                      156
```

```
INPUT NUMBER OF TESTS? 7
INPUT TESTS ALPHANUMERICALLY, ONE AT A TIME? ABCD
? EFGH
? IJKL
? AEI
? BFJ
? CGK
? DHL
THIS IS A MANUAL
THIS IS COHERENT

TYPE IN 1 IF YES
DO YOU WANT TO PRINT OUT EVENTS, CCS, AND PERPS? 1

EVENT                OC                   PERP
*****                **                   ****
                     ABCD                 ABCDEFGHIJKL
D                    ABC                  ABCHL
C                    ABD                  ABDGK
CD                   AB                   AB
B                    ACD                  ACDFJ
BD                   AC                   AC
BC                   AD                   AD
BCD                  A                    A
A                    BCD                  BCDEI
AD                   BC                   BC
AC                   BD                   BD
ACD                  B                    B
AB                   CD                   CD
ABD                  C                    C
ABC                  D                    D
ABCD
                     EFGH                 ABCDEFGHIJKL
H                    EFG                  DEFGL
G                    EFH                  CEFHK
GH                   EF                   EF
F                    EGH                  BEGHJ
FH                   EG                   EG
FG                   EH                   EH
FGH                  E                    E
E                    FGH                  AFGHI
EH                   FG                   FG
EG                   FH                   FH
EGH                  F                    F
EF                   GH                   GH
EFH                  G                    G
EFG                  H                    H
EFGH
```

	IJKL	ABCDEFGHIJKL
L	IJK	DHIJK
K	IJL	CGIJL
KL	IJ	IJ
J	IKL	BFIKL
JL	IK	IK
JK	IL	IL
JKL	I	I
I	JKL	AEJKL
IL	JK	JK
IK	JL	JL
IKL	J	J
IJ	KL	KL
IJL	K	K
IJK	L	L
IJKL		
	AEI	ABCDEFGHIJKL
I	AE	AEJKL
E	AI	AFGHI
EI	A	A
A	EI	BCDEI
AI	E	E
AE	I	I
AEI		
	BFJ	ABCDEFGHIJKL
J	BF	BFIKL
F	BJ	BEGHJ
FJ	B	B
B	FJ	ACDFJ
BJ	F	F
BF	J	J
BFJ		
	CGK	ABCDEFGHIJKL
K	CG	CGIJL
G	CK	CEFHK
GK	C	C
C	GK	ABDGK
CK	G	G
CG	K	K
CGK		
	DHL	ABCDEFGHIJKL
L	DH	DHIJK
H	DL	DEFGL
HL	D	D
D	HL	ABCHL
DL	H	H
DH	L	L
DHL		

```
DO YOU WANT TO PRINT OUT OP PAIRS? 1

OP1                   OP2
***                   ***
BCD                   EI
ACD                   FJ
ABD                   GK
ABC                   HL
ABCD                  EFGH
ABCD                  IJKL
ABCD                  AEI
ABCD                  BFJ
ABCD                  CGK
ABCD                  DHL
FGH                   AI
EGH                   BJ
EFH                   CK
EFG                   DL
EFGH                  IJKL
EFGH                  AEI
EFGH                  BFJ
EFGH                  CGK
EFGH                  DHL
JKL                   AE
IKL                   BF
IJL                   CG
IJK                   DH
IJKL                  AEI
IJKL                  BFJ
IJKL                  CGK
IJKL                  DHL
AEI                   BFJ
AEI                   CGK
AEI                   DHL
BFJ                   CGK
BFJ                   DHL
CGK                   DHL
```

```
INPUT NUMBER OF TESTS? 3
INPUT TESTS ALPHANUMERICALLY, ONE AT A TIME? ABC
? CDE
? ABDE
THIS IS NOT A MANUAL
THIS IS NOT COHERENT

TYPE IN 1 IF YES
DO YOU WANT TO PRINT OUT EVENTS, OCS, AND PERPS? 1

EVENT                   OC                      PERP
*****                   **                      ****
                        ABC                     ABCDE
C                       AB                      ABDE
B                       AC                      ACDE
BC                      A                       A
A                       BC                      BCDE
AC                      B                       B
AB                      C                       CDE
ABC
                        CDE                     ABCDE
E                       CD                      ABCD
D                       CE                      ABCE
DE                      C                       ABC
C                       DE                      ABDE
CE                      D                       D
CD                      E                       E
CDE
                        ABDE                    ABCDE
E                       ABD                     ABCD
D                       ABE                     ABCE
DE                      AB                      ABC
B                       ADE                     ACDE
BE                      AD                      AD
BD                      AE                      AE
BDE                     A                       A
A                       BDE                     BCDE
AE                      BD                      BD
AD                      BE                      BE
ADE                     B                       B
AB                      DE                      CDE
ABE                     D                       D
ABD                     E                       E
ABDE

DO YOU WANT TO PRINT OUT OP PAIRS? 1

OP1                     OP2
***                     ***
C                       DE
BC                      BDE
AC                      ADE
AB                      DE
ABC                     CDE
ABC                     ABDE
C                       AB
CE                      ABE
CD                      ABD
CDE                     ABDE
```

```
DO YOU WANT TO LIST THE COUNTEREXAMPLES TO CONDITION M? 1

X CC Y, Y CP Z, BUT X NOT CC Z
X                   Y                   Z
*                   *                   *
C                   AB                  C
AB                  C                   AB
DE                  C                   DE
C                   DE                  C
DE                  AB                  DE
AB                  DE                  AB
```

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER U.S.N.A. - TSPR; no. 121 (1982)	2. GOVT ACCESSION NO. AD-A124228	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) AN ALGEBRAIC APPROACH TO EMPIRICAL SCIENCE AND QUANTUM LOGIC.		5. TYPE OF REPORT & PERIOD COVERED Final: 1981/1982
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Thomas, Timothy Scott		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS United States Naval Academy, Annapolis.		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS United States Naval Academy, Annapolis.		12. REPORT DATE 29 July 1982
		13. NUMBER OF PAGES 80
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release; Its distribution is UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

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18. SUPPLEMENTARY NOTES

Accepted by the U. S. Tricent Scholar Committee.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Quantum theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper develops some of the work of Foulis, Randall, Aerts, and Piron in the fields of empirical science and quantum logic from an algebraic point of view. More specifically, it begins with three axioms of what is called a "subtraction algebra," and generates various theorems associated with properties which are useful in empicical science.

After a foundation is established, it moves on to define the term

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) "manual," a tool devised by Foulis and Randall in their study. We define it as a "dominated, atomic, semi-Boolean algebra" which satisfies an additional condition called "condition M." Several properties of the manual are discussed, and different types of manuals are given: classical semi-classical and non-classical.

We define operational complements, orerational perspectivity, atoms, events, and tests, before moving on to define a logic, and how it is derived from a manual. properties of the logic are discussed including a subtraction operation, a partial order, and an ortho complement.

Next, a computer program is presented. Its purpose is to take a finite semi-Boolean algebra and decide if the algebra is a manual. This is followed by a brief non-classical probabilistic discussion, which includes topics such as weights, pure states, and dispersion-free states.

Aerts' and Piron's work with properties, states, and questions is briefly discussed before moving on to several examples, some of them arising from navigation problems. The examples include the "hook," the "square," the "Wright Triangle," and the "free algebra." Empirical techniques are demonstrated on these examples. The examples comprise the bulk of this paper.

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